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On generalization of the notion of Moufang loop to n-ary case

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Abstract. Using isotopical approach we generalize concept of binary Moufang loop on n-ary ($n > 2$) case. We give two examples of ternary Moufang loop which are not a ternary group.

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1 Introduction

Previously, let us give basic notions and designations from [2].

1. Given a set Q, let its elements be designated by small latin characters. For short, let $\{x_i\}_{i=m}^{k}$ or $\{x_i\}_m^k$ denote the sequence $x_m, x_{m+1}, \ldots, x_k$. We will often use the designation x_m^k instead of $\{x_i\}_m^k$ if it is clear which index is being changed. The symbol x_m^k makes sense if $m \leq k$. If $m = k$, then x_m^m means simply an element x_m. If $m > k$, then by x_m^k we shall understand empty sequence (empty set). The sequence $a, a, \ldots, a$ (k times) is denoted by $\overset{k}{a}$. The symbol $\overset{0}{a}$ means empty sequence.

Let Q^n be the Cartesian power of the set Q, i.e. Q^n consists of all ordered sequences a_1^n, $a_i \in Q$ ($i = 1, 2, \ldots, n$). A mapping $A : Q^n \to Q$ is called an *n-ary operation*, and one number n is called the *arity* of the operation A. A set Q with n-ary operation A is called an *n-groupoid* and is denoted by $Q(A)$. If an operation A puts into correspondence an element $b \in Q$ to the sequence $a_1^n \in Q^n$, then we write $A(x_1^n) = b$. The operations defined on the set Q are denoted by capital latin characters $A, B, C, \ldots$ or by parenthesis $(a_1^n) = b$.

An n-groupoid $Q(A)$ is called an *n-quasigroup* or *n-ary quasigroup* if in the equality $A(x_1^n) = x_{n+1}$ any n elements of $x_1^{n+1} \in Q$ uniquely define the $(n+1)$-th one. This definition is equivalent to the following one: algebra $Q(A)$ with one n-ary operation A, which is uniquely reversible at each place, is called an n-quasigroup [3, p. 48]. For convenience, the quasigroup operation A itself of n-quasigroup $Q(A)$ as a rule is called a quasigroup too.

If in the n-quasigroup $Q(A)$ there exists at least one element e such that $A\left(\overset{i-1}{e}, x, \overset{n-i}{e}\right) = x$ for any $x \in Q$ and any $i = 1, 2, \ldots, n$, then $Q(A)$ is called an *n-loop* with an identity element e.

2. A quasigroup B is called an *isotope* of a quasigroup A (A and B have equal arity n and are defined on the same set Q) if there exists a sequence $T = (\alpha_1^{n+1})$ of substitutions of the set Q such that $B(x_1^n) = \alpha_{n+1}^{-1} A(\{\alpha_i x_i\}_1^n)$ for all $x_1^n \in Q^n$.

The denotation is: $B = A^T$. The sequence T is called an *isotopy.*

A substitution α_i is called an *i-th component* of the isotopy $T = (\alpha_i^{n+1})$.

From $C = B^T, B = A^S \Rightarrow C = A^{ST}$, where ST is the product of isotopies: $P = ST = (\alpha_1^{n+1})(\beta_1^{n+1}) = (\{\alpha_i \beta_i\}_1^{n+1})$;

From $B = A^S \Rightarrow A = B^{S^{-1}}$. An isomorphism, i.e. an isotopy of the form $(\overset{n+1}{\alpha})$, is a particular case of the isotopy. The principal isotopy is another particular case of isotopy: an isotopy of the form $S = (\alpha_1^n, \varepsilon)$ is called the *principal* one, and one quasigroup $B = A^S$ is called the *principal isotope* of the quasigroup A (ε is an identical substitution).

Theorem 1. *If a quasigroup B is isotopic to a quasigroup A, then it is isomorphic to some its principal isotope:*

$$B = A^T, T = (\alpha_1^{n+1}) \Rightarrow B = (A^S)^{\alpha_{n+1}}.$$

Among the principal isotopes there are LP-isotopes that stand out. The notion of n-quasigroup translation is necessary for their definition.

Let us designate the sequence $a_1^n \in Q^n$ by $\bar{a}$, the sequence $\{a_1^{i-1}, a_{i+1}^n\}$ by ${}_i(\bar{a})$. Such a sequence is called *i-section* of the sequence $\bar{a} = \{a_1^n\}$. Let an n-quasigroup A be defined on Q. Then a mapping $L_i(\bar{a})$, defined by the equality

$$L_i(\bar{a})x = A(a_1^{i-1}, x, a_{i+1}^n),$$

is called an *i-translation* relative to the sequence $\bar{a}$ or translation relative to i-section of the sequence $\bar{a}$. By virtue of the n-quasigroup definition, i-translations are substitutions of the set Q for any $\bar{a} \in Q^n$ and for any $i = 1, 2, \ldots, n$. Notice that $L_1(\bar{a})x = A(x, a_2^n), L_n(\bar{a})x = A(a_1^{n-1}, x)$; in the case when $n = 2$ we have $\bar{a} = \{a_1, a_2\} \in Q^2$, $L_1(\bar{a})x = A(x, a_2)$ is right translation, $L_2(\bar{a})x = A(a_1, x)$ is left translation of the quasigroup A.

Principal isotope $B = A^T$ of a quasigroup A, where $T = (\alpha_1^n, \varepsilon)$ and $\alpha_i = L_i^{-1}(\bar{a})$, is called an *$LP$-isotope* of the quasigroup A.

Theorem 2. *Every LP-isotope of a quasigroup is a loop.*

Proof. Indeed, let us be given an n-quasigroup $Q(A)$. We consider its LP-isotope $B(x_1^n) = A^T(x_1^n)$, where $T = (\alpha_1^n, \varepsilon)$, $\bar{a} = a_1^n$.

Let us show that $e = A(a_1^n)$ is an identity element of the quasigroup $Q(B)$. Notice that $L_i(\bar{a})a_i = A(a_1^n)$. Let $A(a_1^n) = e$. Then $\alpha_i e = L_i^{-1}(\bar{a})e = a_i$. Therefore,

$$B(\overset{i-1}{e}, x, \overset{n-i}{e}) = A^T(\overset{i-1}{e}, x, \overset{n-i}{e}) = A(\{\alpha_j e\}_{j=1}^{i-1}, \alpha_i x, \{\alpha_j e\}_{j=i+1}^n) =$$

$$= A(a_1^{i-1}, \alpha_i x, a_{i+1}^n) = L_i(\bar{a})(\alpha_i x) = L_i(\bar{a})(L_i^{-1}(\bar{a})x) = x,$$

i.e., e is the identity element of the loop $Q(B)$. $\square$

Theorem 3. *If a loop is principally isotopic to an n-quasigroup, then it is LP-isotopic to this quasigroup.*

Proof. Indeed, let A be an n-quasigroup, $B(x_1^n) = A^T(x_1^n)$, $T = (\alpha_1^n, \varepsilon)$ and let $Q(B)$ be a loop with identity element e. Then $x = B(\overset{i-1}{e}, x, \overset{n-i}{e}) = A^T(\overset{i-1}{e}, x, \overset{n-i}{e}) = A(\{\alpha_j e\}_{j=1}^{i-1}, \alpha_i x, \{\alpha_j e\}_{j=i+1}^{n})$.

Let us assume that $\alpha_i e = a_i$, $i \in \{1, 2, \ldots, n\}$, $\bar{a} = a_1^n$. Therefore, $x = A(a_1^{i-1}, \alpha_i x, a_{i+1}^n) = L_i(\bar{a})\alpha_i x$, whence $\alpha_i = L_i^{-1}(\bar{a})$, hence $B = A^T$ is an LP-isotope of the quasigroup A. □

3. Let $Q(A)$ be an n-quasigroup. From the definition of translation $L_i(\bar{a})$ of the quasigroup $Q(A)$ we have that in this quasigroup the following identity holds:

$$L_i(\bar{a})x = A(a_1^{i-1}, x, a_{i+1}^n), \tag{1}$$

where x runs through all the set Q, $L_i(\bar{a})$ is a substitution of the set Q for $\forall \bar{a} = x_1^n \in Q^n$ and for $\forall i = 1, 2, \ldots, n$.

From the identity (1) it results that with respect to the n-quasigroup $Q(A)$ the following identities hold:

$$L_i^{-1}(\bar{a})A(a_1^{i-1}, x, a_{i+1}^n) = A(a_1^{i-1}, L_i^{-1}(\bar{a})x, a_{i+1}^n); \tag{2}$$
$$L_i(\bar{a})A(a_1^{i-1}, x, a_{i+1}n) = A(a^{i-1}L_i(\bar{a})x, a_{i+1}^n). \tag{3}$$

Indeed, by replacing in the identity (1) $x \to L_i^{-1}(\bar{a})x$, we get the following identity:

$$x = A(a_1^{i-1}, L_i^{-1}(\bar{a})x, a_{i+1}^n). \tag{4}$$

On the other hand, applying the substitution $L_i^{-1}(\bar{a})$ from left to the identity (1), we get the following identity:

$$x = L_i(\bar{a})A(a_1^{i-1}, x, a_{i+1}^n). \tag{5}$$

From the identities (5) $\wedge$ (4) it follows the identity (2); by replacing $x \to L_i(\bar{a})x$ in the identity (2), and then by applying the substitution $L_i(\bar{a})$ to the obtained identity, it follows the identity (3).

From (1) by replacing $x \to L_i^{-2}(\bar{a})$, evidently the following identity results:

$$L_i^{-1}(\bar{a})x = A(a_1^{i-1}, L_i^{-2}(\bar{a})x, a_{i+1}^n). \tag{6}$$

By definition, LP-isotope $Q(B)$ of n-quasigroup $Q(A)$ relative to the sequence $\bar{a} = a_1^n \in Q^n$ is a principal isotope of this quasigroup of the form

$$B(x_1^n) = A(\{L_i^{-1}(\bar{a})x_i\}_{i=1}^n). \tag{7}$$

The LP-isotope $Q(B)$ of the quasigroup $Q(A)$ is a loop with the identity element $e = A(a_1^n)$ [2, p. 13].

4. For loops (with binary operation) the notion of an IP-loop (loop with reversibility) is defined: a loop $Q(\cdot)$ is called a *IP-loop* if for any $a, b, \in Q$ the following holds: ${}^{-1}a(ab) = b$, $(ba)a^{-1} = b$, where ${}^{-1}aa = aa^{-1} = 1$.

2 Some results

In [3, p. 48] it is noted that the main research object is not IP-loops, but Moufang loops, which is a narrower class. Namely, a loop is called a *Moufang loop* if all the loops which are isotopic to it are the IP-loops. The following theorem is true:
Moufang Theorem: *A loop $Q(\cdot)$ is Moufang if and only if the following identity holds:*

$$(xy)(zx) = [x(yz)]x.$$

Let us note that the Moufang loop is also equivalently defined by one of the following identities:

$$x(y \cdot xz) = (xy \cdot x)z,$$

$$(zx \cdot y)x = z(x \cdot yx) \quad [1, \text{p.}59].$$

In [1] the notions of the loop with the property of reversibility (IP-loop) and Moufang loop in the context of more general notion of IP-quasigroup are studied in detail. In [2] the generalisations of the notions of IP-loop and Moufang loop are also considered within the more general notion of IP-n-quasigroup. The notion of IP-n-loop admits the following

Definition 1. An n-loop $Q(A)$ is called an *n-loop with the property of reversibility* (or *IP-n-loop*) if there exists the system of substitutions ν_{ij} $(i, j = 1, 2, \ldots, n)$ of the set Q (with $\nu_{ij} = \varepsilon$ being the identical substitution) such that the following identities hold:

$$A(\{\nu_{ij}x_j\}_{j=1}^{i-1}, A(x_1^n), \{\nu_{ij}x_j\}_{j=i+1}^{n}) = x_i \tag{8}$$

for any $x_i \in Q\,(i = 1, 2, \ldots, n)$. The matrix $\|\nu_{ij}\|$ is called an *inversion matrix* of IP-loop $Q(A)$, and substitutions ν_{ij} are called *inversion substitutions* [2, p.66].

Let us extend without changes the definition of the notion of Moufang loop (binary) given in [3, pp. 18] to n-ary case of the loop. Thus, in the set of all IP-n-loops, a narrower class of n-loops Moufang is singled out, which conforms to the following definition:

Definition 2. An n-loop $Q(A)$ is called a *Moufang n-loop* (or *Moufang loop*) if all the loops isotopic to it are n-loops with the property of reversibility (IP-n-loops).

The following theorem is true:

Theorem 4. *An n-loop $Q(A)$ is a Moufang n-loop if and only if the following condition is met: for any LP-isotope of n-loop $Q(A)$ there exists a system of substitutions $\tilde{\nu}_{ij}$ $(i, j = 1, 2, \ldots, n)$ of the set Q, with $\tilde{\nu}_{ij} = \varepsilon$, such that the following identities are true:*

$$C(\{\tilde{\nu}_{ij}x_j\}_{j=1}^{i-1}, C(x_1^n), \{\tilde{\nu}_{ij}x_j\}_{j=i+1}^{n}) = x_i \tag{9}$$

for any $x_i \in Q$ $(i = 1, 2, \ldots, n)$.

Proof. Necessity. Let $Q(A)$ be a Moufang n-loop. According to Definition 2, all its LP-isotopes $Q(C)$ are IP-n-loops. Thus, according to Definition 1, for any LP-isotope $Q(C)$ of the loop $Q(A)$ the identities (9) are true.

Sufficiency. Let $Q(A)$ be an n-loop and any its LP-isotope $Q(C)$ satisfies identities (9). According to the known theorem: if a loop C is isotopic to an n-loop A, then it is isomorphic to some of its principal isotope:

$$C = A^T, T = (\alpha_1^{n+1}) \Rightarrow C = (A^S)^{\alpha_{n+1}};$$

this principal isotope A^S is a loop. From Theorem 3 the following corollary obviously follows: if a loop C_1 is principally isotopic to an n-loop A, then it is LP-isotopic to the loop A: $C_1 = A^S, S = (\beta_1^n, \varepsilon) \Rightarrow \beta_i = L_i^{-1}(\bar{a})$.

By virtue of these theorems, it follows that any loop $Q(C)$ which is isotopic to an n-loop $Q(A)$ is isomorphic to some LP-isotope of the loop $Q(A)$. Since, according to the theorem's condition, all LP-isotopes of the loop $Q(A)$ are IP-loops, then any loop which is isotopic to n-loop $Q(A)$ is an IP-n-loop, i.e. $Q(A)$ is a Moufang n-loop. The theorem is proved. $\square$

Let Moufang n-loop $Q(A)$ be set by identities (9). Applying formula (7) to identities (9) we get that identities (9) become as follows:

$$A(\{L_j^{-1}(\bar{a})\tilde{\nu}_{ij}x_j\}_{j=1}^{i-1}, L_i^{-1}(\bar{a})A(\{L_j^{-1}(\bar{a})x_j\}_{j=1}^{n}), \{L_j^{-1}(\bar{a})\tilde{\nu}_{ij}x_j\}_{j=i+1}^{n}) = x_i \qquad (10)$$

for any $x_i \in Q$ $(i = 1, 2, \ldots, n)$, where $L_i(\bar{a})$ is an i-translation relative to any $\bar{a} = a_1^n \in Q^n$.

Let us call identities (10) and the equivalent to them ones as *identities of Moufang n-loop* $Q(A)$. By replacing $x_j \to L_j(\bar{a})x_j$ $(j = 1, 2, \ldots, n)$ in (10), we get the following identities:

$$A(\{L_j^{-1}(\bar{a})\tilde{\nu}_{ij}L_j(\bar{a})x_j\}_{j=1}^{i-1}, L_i^{-1}(\bar{a})A(x_1^n), \{L_j^{-1}(\bar{a})\tilde{\nu}_{ij}L_j(\bar{a})x_j\}_{j=i+1}^{n}) = L_i(\bar{a})x_i \quad (11)$$

for any $x_i \in Q$ $(i = 1, 2, \ldots, n)$ and $\bar{a} = a_1^n \in Q^n$.

Let a Moufang n-loop be set by identities (11). In particular case when $n = 2$, identities (11) are equivalent to the following system of two identities:

$$\begin{aligned} A(L_1^{-1}(\bar{a})A(x_1^2), L_2^{-1}(\bar{a})\tilde{\nu}_{12}L_2(\bar{a})x_2) &= L_1(\bar{a})x_1, \\ A(L_1^{-1}(\bar{a})\tilde{\nu}_{21}L_1(\bar{a})x_1, L_2^{-1}(\bar{a})A(x_1^2)) &= L_2(\bar{a})x_2 \end{aligned} \qquad (12)$$

for any $x_1, x_2 \in Q$, any $\bar{a} = \{a_1, a_2\} \in Q^2$, where ν_{12}, ν_{21} are the inversion substitutions of IP-n-loop $Q(B)$ which is LP-isotopic to the loop $Q(A)$. By the definition of translation in n-quasigroup $Q(A)$ it follows that in the particular case with $n = 2$ the following relations take place:

$$\bar{a} = \{a_1, a_2\}, L_1\bar{a})x = A(x, a_2) \ - \ \text{right translation},$$

$$L_2(\bar{a})x = A(a_1, x) \text{ is left translation of the quasigroup } Q(A),$$

where x runs through the whole set Q. Let us denote $A = (\cdot)$. Then these relations can be written in the following form:

$$L_1(\bar{a}) = R_{a_2}, L_2(\bar{a}) = L_{a_1}, \tag{13}$$

from which it results that $L_1^{-1}(\bar{a}) = R_{a_2}^{-1}, L_2^{-1}(\bar{a}) = L_{a_1}^{-1}$.

Then, in view of the relations (13), identities (12) become as follows:

$$\begin{gathered} R_{a_2}^{-1}(x_1 \cdot x_2) \cdot L_{a_1}^{-1}\tilde{\nu}_{12}L_{a_1}x_2 = R_{a_2}x_1, \\ R_{a_2}^{-1}\tilde{\nu}_{21}R_{a_2}x_1 \cdot L_{a_1}^{-1}(x_1 \cdot x_2) = L_{a_1}x_2 \end{gathered} \tag{14}$$

for any $x_1, x_2, a_1, a_2 \in Q$.

Let us consider the second identity from (14). With $x_2 = L_{x_1}^{-1}a_1$ it entails the following identity:

$$R_{a_2}^{-1}\tilde{\nu}_{21}R_{a_2}x_1 = a_1 L_{x_1}^{-1}a_1. \tag{15}$$

By replacing expression $R_{a_2}^{-1}\tilde{\nu}_{21}x_1$ in (14) by identically equal to it expression from identity (15), we get the following identity:

$$(a_1 \cdot L_{x_1}^{-1}a_1) \cdot L_{a_1}^{-1}(x_1 \cdot x_2) = a_1 x_2 \tag{16}$$

for any $a_1, x_1, x_2 \in Q$. So, since the loop $Q(\cdot)$ (binary), i.e. $Q(A)$ set by identities (14), is an IP-loop, then, as it is known, it possesses the following properties:

$$^{-1}x = x^{-1},\ (x \cdot y)^{-1} = y^{-1} \cdot x^{-1},\ L_x^{-1} = L_{x^{-1}},\ R_x^{-1} = R_{x^{-1}},$$

where L_x, R_x respectively are left and right translations of the loop $Q(\cdot)$ relative to an arbitrary element $x \in Q$, and x^{-1} is a right inverse element for x: $x \cdot x^{-1} = 1$, 1 is an identity element of the loop $Q(\cdot)$; $(x^{-1})^{-1} = x$. Therefore, identity (16) gets the following form:

$$(a_1 \cdot x_1^{-1}a_1) \cdot a_1^{-1}(x_1 \cdot x_2) = a_1 x_2$$

for any $a_1, x_1, x_2 \in Q$.

Renaming variables in this identity as follows: $a_1 \to x, x_1 \to y, x_2 \to z$, we get the following identity:

$$x(y^{-1} \cdot x) \cdot (x^{-1} \cdot yz) = xz \Longleftrightarrow x^{-1} \cdot yz = (x^{-1}y)x^{-1} \cdot xz.$$

Replacing in the last identity $y \to x \cdot yx = L_x R_x y$ we get the identity:

$$x^{-1}[(x \cdot yx)z] = [x^{-1}(x \cdot yx)]x^{-1} \cdot xz \Longleftrightarrow x^{-1}[(x \cdot yx)z] = y \cdot xz \Longleftrightarrow (x \cdot yx)z = x(y \cdot xz)$$

– the identity of left Bol loop.

And since $Q(\cdot)$ is an IP-loop and at the same time it is a left Bol loop, then, by the known theorem, $Q(\cdot)$ is a Moufang loop assignable by the following identity:

$$x(y \cdot xz) = (xy \cdot x)z \Longleftrightarrow (xy)(zx) = [x(yz)]x.$$

By analogy, the first identity from (14) entails the same identities.

With $n = 3$ an operation is called *ternary*, and identities (10) become identities of ternary Moufang loop (Moufang 3-loop) $Q(A)$ and have the following compact writing:

$$A(\{L_j^{-1}(\bar{a})\tilde{\nu}_{ij}x_j\}_{j=1}^{i-1}, L_i^{-1}(\bar{a})A(\{L_j^{-1}(\bar{a})x_j\}_{j=1}^{3}), \{L_j^{-1}(\bar{a})\tilde{\nu}_{ij}x_j\}_{j=i+1}^{3}) = x_i \qquad (17)$$

for any $x_i \in Q$ $(i = 1, 2, 3)$ and $\forall \bar{a} = a_1^3 \in Q^3$, where $\tilde{\nu}_{i1}, \tilde{\nu}_{i2}, \tilde{\nu}_{i3}$ are inversion substitutions of the IP-n-loop $Q(B)$ which is LP-isotopic to the loop $Q(A)$, and $L_i(\bar{a})$ is an i-translation relative to $\bar{a}$.

Let us note one of the main properties of Moufang n-loop in the form of the following theorem:

Theorem 5. *Any loop which is isotopic to Moufang n-loop is also a Moufang n-loop.*

This theorem is proved in the context of broader generalisation of the notion of Moufang loop [2, p. 75].

3 Example of a ternary noncommutative Moufang loop

The following example demonstrates the existence of 3-ary Moufang loops which differ from 3-groups. Let $K(+, \cdot)$ be an associative (not necessary commutative) ring with unity which has characteristic 3, i.e. there exists such a positive integer n that for every element $x \in K$ the equality $n \cdot x = \underbrace{x + \ldots + x}_{n \text{ times}} = 0$ holds, with the least such number $p = 3$ for which $3 \cdot x = x + x + x = 0$ for every $x \in K$, and let $K'(\cdot)$ be an abelian subgroup in a multiplicative semigroup $K(\cdot)$ of the ring K, consisting not only from 1 and such that the mapping $x \to s \cdot x$ is a substitution of the set K for any $s \in K'$ with $s^2 = 1$ for $\forall s \in K'$ (in particular, $K = Z_3$ is a ring of residue classes modulo 3).

Let us consider Cartesian product $Q = K' \times K = \{\langle s, x\rangle | s \in K' \wedge x \in K\}$ of the sets K' and K, and also the Cartesian 3-rd degree $Q^3 = \{\langle s_i, x_i\rangle_1^3 | s_i \in K', x_i \in K\}$ of the set Q. Let us denote by $\bar{\bar{a}}$ the sequence $\bar{\bar{a}} = \langle r_i, a_i\rangle_{i=1}^3 \in Q^3$. Let us define ternary operation

$$A(\langle s_1, x_1\rangle, \langle s_2, x_2\rangle, \langle s_3, x_3\rangle) = \langle s_1 s_2 s_3, s_2 x_1 + s_3 x_2 + s_1 x_3\rangle \qquad (18)$$

for $\forall s_1, s_2, s_3 \in K'$ and $\forall x_1, x_2, x_3 \in K$ on the set $Q = K' \times K$ of ordered pairs of the form $\langle s, x\rangle \in Q$. Let us define the following mapping to this operation:

$$\nu_{ij} : Q \to Q, \quad \nu_{ij}\langle s_j, x_j\rangle = \begin{cases} \langle s_j, -s_j x_j\rangle & \text{when } j \neq i, \\ \langle s_j, x_j\rangle & \text{when } j = i \end{cases} \qquad (19)$$

for $\forall s_j \in K'$, $\forall x_j \in K$ $(j = 1, 2, 3)$ and for $\forall i = 1, 2, 3$. It is obvious that ν_{ij} is a substitution of the set $Q = K \times K'$.

It is easy to verify that 3-groupoid $Q(A)$ with operation (18) is a ternary loop (3-loop) with unity $< 1, 0 >$. Let us prove that this loop $Q(A)$ is a required Moufang 3-loop.

Really, according to the definition of LP-isotope of n-quasigroup, for a loop $Q(A)$ defined on the set $Q = K' \times K$ by formula (18) and for arbitrary sequence $\bar{\bar{a}} = \langle r_i, a_i \rangle_{i=1}^3 \in Q^3$ the specific LP-isotope $Q(B)$ of the loop $Q(A)$ is appropriately formed according to the following formula:

$$B(\langle s_i, x_i \rangle_{i=1}^3) = A(\{L_i^{-1}(\bar{\bar{a}})\langle s_i, x_i \rangle\}_{i=1}^3) \tag{20}$$

for $\forall s_i \in K'$, $\forall x_i \in K$ $(i = 1, 2, 3)$ and for arbitrary sequence $\bar{\bar{a}} \in Q^3$. Let us fix arbitrary sequence and with this restriction consider the respective LP-isotope $Q(B)$ of the loop $Q(A)$ with the operation (18).

From the definition of i-translation of quasigroup $L_i(\bar{a})$ it follows that $L_i(\bar{\bar{a}})$ is a substitution of the set $Q = K' \times K$ for every $i = 1, 2, 3$. Therefore $L_i^{-1}(\bar{\bar{a}})\langle s_i, x_i \rangle$ is some ordered pair of the form $\langle s, x \rangle$ from $Q = K' \times K$, i.e.

$$L_i^{-1}(\bar{\bar{a}})\langle s_i, x_i \rangle = \langle t_i, y_i \rangle \tag{21}$$

for every $i = 1, 2, 3$. Let us find the explicit form of the pair $\langle t_i, y_i \rangle$, expressed through s_i, x_i. According to the definition of i-translation of quasigroup, the following identities are equivalent:

$$(21) \Leftrightarrow \langle s_i, x_i \rangle = L_i(\bar{\bar{a}})\langle t_i, y_i \rangle \Leftrightarrow \langle s_i, x_i \rangle = A((\bar{\bar{a}})_1^{i-1}, \langle t_i, y_i \rangle, (\bar{\bar{a}})_{i+1}^3)$$

for every $i = 1, 2, 3$. The last identity is equivalent to the following system of three identities:

$$\begin{array}{rll} \text{When} \quad i = 1 \Rightarrow 1^\circ. & \langle s_1, x_1 \rangle = & A(\langle t_1, y_1 \rangle, \langle r_2, b_2 \rangle, \langle r_3, a_3 \rangle); \\ \text{when} \quad i = 2 \Rightarrow 2^\circ. & \langle s_2, x_2 \rangle = & A(\langle r_1, a_1 \rangle, \langle t_2, a_2 \rangle, \langle r_3, a_3 \rangle); \\ \text{when} \quad i = 3 \Rightarrow 3^\circ. & \langle s_3, x_3 \rangle = & A(\langle r_1, a_1 \rangle, \langle r_2, a_2 \rangle, \langle t_3, y_3 \rangle). \end{array} \tag{22}$$

When applying formula (18) to identities (22), these identities become as follows:

$$\begin{array}{rcl} 1^\circ.\langle s_1, x_1 \rangle & = & \langle t_1 r_2 r_3, r_2 y_1 + r_3 a_2 + t_1 a_3 \rangle, \\ 2^\circ.\langle s_2, x_2 \rangle & = & \langle r_1 t_2 r_3, t_2 a_1 + r_3 y_2 + r_1 a_3 \rangle, \\ 3^\circ.\langle s_3, x_3 \rangle & = & \langle r_1 r_2 t_3, r_2 a_1 + t_3 a_2 + r_1 y_3 \rangle \end{array}$$

for $\forall s_i, r_i, t_i \in K'$ and $\forall x_i, a_i, y_i \in K$ $(i = 1, 2, 3)$, from which the following equalities result:

$$\begin{array}{rr} & t_1 = r_2 r_3 s_1, \quad t_2 = r_1 r_3 s_2, \quad t_3 = r_1 r_2 s_3, \\ y_1 = r_2(x_1 - r_3 a_2 - r_2 r_3 s_1 a_3), & y_2 = r_3(x_2 - r_1 r_3 s_2 a_1 - r_1 a_3), \\ & y_3 = r_1(x_3 - r_2 a_1 - r_1 r_2 s_3 a_2). \end{array}$$

Therefore,

$$\begin{aligned} \langle t_1, y_1\rangle &= \langle r_2r_3s_1, r_2(x_1 - r_3a_2 - r_2r_3s_1a_3)\rangle,\\ \langle t_2, y_2\rangle &= \langle r_1r_3s_2, r_3(x_2 - r_1r_3s_2a_1 - r_1a_3)\rangle,\\ \langle t_3, y_3\rangle &= \langle r_1r_2s_3, r_1(x_3 - r_2a_1 - r_1r_2s_3a_2)\rangle. \end{aligned} \tag{23}$$

In view of identities (21) and (23) it follows that identity (20) equivalently transforms in the following way:

$$\begin{aligned} B(\langle s_1, x_1\rangle, \langle s_2, x_2\rangle, \langle s_3, x_3\rangle) = \quad & A(L_1^{-1}(\bar{\bar{a}})\langle s_1, x_1\rangle, L_2^{-1}(\bar{\bar{a}})\langle s_2, x_2\rangle, L_3^{-1}(\bar{\bar{a}})\langle s_3, x_3\rangle)\\ \Leftrightarrow B(\langle s_1, x_1\rangle, \langle s_2, x_2\rangle, \langle s_3, x_3\rangle) = \quad & A(\langle r_2r_3s_1, r_2(x_1 - r_3a_2 - r_2r_3s_1a_3)\rangle, \qquad (24)\\ \langle r_1r_3s_2, r_3(x_2 - r_1r_3s_2a_1 - r_1a_3)\rangle, \quad & \langle r_1r_2s_3, r_1(x_3 - r_2a_1 - r_1r_2s_3a_2)\rangle). \end{aligned}$$

Applying formula (18) to the right part of the last identity in (24), we get the following identity:

$$\begin{aligned} B(\langle s_1, x_1\rangle, \langle s_2, x_2\rangle, \langle s_3, x_3\rangle) \;&=\; \langle r_2r_3s_1 \cdot r_1r_3s_2 \cdot r_1r_2s_3,\\ r_1r_3s_2 \cdot r_2(x_1 - r_3a_2 - r_2r_3s_1a_3) \;&+\; r_1r_2s_3 \cdot r_3(x_2 - r_1r_3s_2a_1 - r_1a_3) +\\ r_2r_3s_1 \cdot r_1(x_3 - r_2a_1 - r_1r_2s_3a_2)\rangle & \end{aligned}$$

or

$$\begin{aligned} B(\langle s_1, x_1\rangle, \langle s_2, x_2\rangle, \langle s_3, x_3\rangle) \;&=\; \langle s_1s_2s_3, r_1r_2r_3 \cdot [s_2(x_1 - r_3a_2 - r_2r_3s_1a_3) +\\ s_3(x_2 - r_1r_3s_2a_1 - r_1a_3) \;&+\; s_1(x_3 - r_2a_1 - r_1r_2s_3a_2)]\rangle \end{aligned} \tag{25}$$

for any $s_i, r_i \in K'$, $\forall x_i, a_i \in K$ $(i = 1, 2, 3)$ and for arbitrary fixed sequence $\bar{\bar{a}} = \langle r_i, a_i\rangle_{i=1}^3 \in Q^3$, where $Q = K' \times K$.

Thus, if a 3-loop $Q(A)$ is defined on the set $Q = K' \times K$ by formula (18), then its arbitrary LP-isotope $Q(B)$ can be defined by the formula (25). According to the proof of the Theorem 2, this LP-isotope $Q(B)$ is a loop with identity element:

$$e = A(\langle r_i, a_i\rangle_{i=1}^3) = \langle r_1r_2r_3, r_2a_1 + r_3a_2 + r_1a_3\rangle.$$

Let us show that for a 3-loop $Q(A)$, defined by formula (18), its LP-isotope $Q(B)$ which can be defined by formula (25) is an IP-loop.

Really, according to Definition 1, in order for a 3-loop $Q(B)$, defined on the set $Q = K' \times K$, to be a JP-loop, it is sufficient that the following condition to be met: there exists a system of substitutions $\tilde{\nu}_{ij}$ $(i, j = 1, 2, 3)$ of the set Q, with $\tilde{\nu}_{ij} = \varepsilon$, such that the following identities hold:

$$B(\{\tilde{\nu}_{ij}\langle s_j, x_j\rangle\}_{j=1}^{i-1}, B(\langle s_j, x_j\rangle_{j=1}^3), \{\tilde{\nu}_{ij}\langle s_j, x_j\rangle\}_{j=i+1}^3) = \langle s_i, x_i\rangle \tag{26}$$

for $\forall s_j \in K'$, $\forall x_j \in K$, and for every $i = 1, 2, 3$.

Let us consider mappings:

$$\tilde{\nu}_{ij} : Q \to Q, \quad \tilde{\nu}_{ij}\langle s_j, x_j\rangle = \begin{cases} \langle s_j, -r_1r_2r_3s_j \cdot x_j + c_j\rangle, & \text{when } \; j \neq i,\\ \langle s_j, x_j\rangle, & \text{when } \; j = i \end{cases} \tag{27}$$

for $\forall s_j \in K'$, $\forall x_j \in K$ $(j = 1, 2, 3)$, and for $\forall i = 1, 2, 3$ with fixed arbitrary sequence $\bar{\bar{a}} = \langle r_i, a_i \rangle \in Q^3$, where c_j are some elements from K which are to be determined.

Obviously, this mapping is a substitution of the set $Q = K' \times K$.

Let us prove that there exist such elements $c_j \in K$ $(j = 1, 2, 3)$ that substitutions (27) satisfy identities (26).

At first, suppose that there already exist such elements $c_i \subset Q$ $(i = 1, 2, 3)$ that the substitutions $\tilde{\nu}_{ij}$ $(j = 1, 2, 3)$ of the set Q, defined by the formula (27), satisfy identities (26). Identities (26) are equivalent to the following system of three identities:

$$\begin{cases} \text{When} \quad i = 1 \Rightarrow \quad I.\ B(B(\langle s_j, x_j \rangle_{j=1}^3), \tilde{\nu}_{12}\langle s_2, x_2 \rangle, \tilde{\nu}_{13}\langle s_3, x_3 \rangle) = \langle s_1, x_1 \rangle, \\ \text{when} \quad i = 2 \Rightarrow \quad II.\ B(\tilde{\nu}_{21}\langle s_1, x_1 \rangle, B(\langle s_j, x_j \rangle_{j=1}^3), \tilde{\nu}_{23}\langle s_3, x_3 \rangle) = \langle s_2, x_2 \rangle, \\ \text{when} \quad i = 3 \Rightarrow \quad III.\ B(\tilde{\nu}_{31}\langle s_1, x_1 \rangle, \tilde{\nu}_{32}\langle s_2, x_2 \rangle, B(\langle s_j, x_j \rangle_{j=1}^3)) = \langle s_3, x_3 \rangle. \end{cases} \tag{28}$$

Let us denote by F the second component of the pair in the right part of identity (25):

$$F = r_1 r_2 r_3 \cdot [s_2(x_1 - r_3 a_2 - r_2 r_3 s_1 a_3) + s_3(x_2 - r_1 r_3 s_2 a_1 - r_1 a_3) + s_1(x_3 - r_2 a_1 - r_1 r_2 s_3 a_2)]. \tag{29}$$

Applying formulae (25), (27), and (29) to the left sides of identities (28), we get the equivalent system of three identities:

$$\begin{cases} I.\, B(\langle s_1 s_2 s_3, F \rangle, \langle s_2, -r_1 r_2 r_3 s_2 \cdot x_2 + c_2 \rangle, \langle s_3, -r_1 r_2 r_3 s_3 \cdot x_3 + c_3 \rangle) = \langle s_1, x_1 \rangle; \\ II.\, B(\langle s_1, -r_1 r_2 r_3 s_1 \cdot x_1 + c_1 \rangle, \langle s_1 s_2 s_3, F \rangle, \langle s_3, -r_1 r_2 r_3 s_3 \cdot x_3 + c_3 \rangle) = \langle s_2, x_2 \rangle; \\ III.\, B(\langle s_1, -r_1 r_2 r_3 s_1 \cdot x_1 + c_1 \rangle, \langle s_2, -r_1 r_2 r_3 s_2 \cdot x_2 + c_2 \rangle, \langle s_1 s_2 s_3, F \rangle) = \langle s_3, x_3 \rangle. \end{cases} \tag{30}$$

Applying now the same formula (25) to the left sides of identities (30), but relative to new components of the operation B (i.e. considering new components as variables $S_1, X_1; S_2, X_2; S_3, X_3$ respectively), we get the equivalent system of identities:

$$\begin{cases} I.\, \langle s_1s_2s_3 \cdot s_2 \cdot s_3, r_1r_2r_3 \cdot [s_2(F - r_3a_2 - r_2r_3s_1s_2s_3a_3) + \\ \quad + s_3(-r_1r_2r_3s_2 \cdot x_2 + c_2 - r_1r_3s_2a_1 - r_1a_3) + \\ \quad + s_1s_2s_3(-r_1r_2r_3s_3 \cdot x_3 + c_3 + c_1 - r_3a_2 - r_1r_2s_3a_2)]\rangle = \langle s_1, x_1\rangle, \\ II.\, \langle s_1 \cdot s_1s_2s_3 \cdot s_3, r_1r_2r_3 \cdot [s_1s_2s_3(-r_1r_2r_3s_1 \cdot x_1 - r_3a_2 - r_2r_3s_1a_3) + \\ \quad + s_3(F - r_1r_3s_1s_2s_3a_1 - r_1a_3) + \\ \quad + s_1(-r_1r_2r_3s_3 \cdot x_3 + c_3 - r_2a_1 - r_1r_2s_3a_2)]\rangle = \langle s_2, x_2\rangle, \\ III.\, \langle s_1 \cdot s_2 \cdot s_1s_2s_3, r_1r_2r_3 \cdot [s_2(-r_1r_2r_3s_1 \cdot x_1 + c_1 - r_3a_2 - r_2r_3s_1a_3) + \\ \quad + s_1s_2s_3(-r_1r_2r_3s_2 \cdot x_2 + c_2 - r_1r_3s_2a_1 - r_1a_3) + \\ \quad + s_1(F - r_2a_1 - r_1r_2s_1s_2s_3a_2)]\rangle = \langle s_3, x_3\rangle. \end{cases} \tag{31}$$

Multiplying from left by $r_1r_2r_3s_3$, $r_1r_2r_3s_1$, $r_1r_2r_3s_2$ the second components of the identities I, II, III of the identities system (31) respectively, we get the following system of three identities:

$$\begin{cases} I.\, s_2s_3 \cdot (F - r_3a_2 - r_2r_3s_1s_2s_3a_3) + \\ \quad +1 \cdot (-r_1r_2r_3s_2 \cdot x_2 + c_2 - r_1r_3s_2a_1 - r_1a_3) + \\ \quad + s_1s_2 \cdot (-r_1r_2r_3s_3 \cdot x_3 + c_3 - r_2a_1 - r_1r_2s_3a_2) = r_1r_2r_3s_3 \cdot x_1, \\ II.\, s_2a_3 \cdot (-r_1r_2r_3s_1 \cdot x_1 + c_1 - r_3a_2 - r_2r_3s_1a_3) + \\ \quad + s_1s_3 \cdot (F - r_1r_3s_1s_2s_3a_1 - r_1a_3) + \\ \quad +1 \cdot (-r_1r_2r_3s_3 \cdot x_3 + c_3 - r_2a_1 - r_1r_2s_3a_2) = r_1r_2r_3s_1 \cdot x_2, \\ III.\, 1 \cdot (-r_1r_2r_3s_1 \cdot x_1 + c_1 - r_3a_2 - r_2r_3s_1a_3) + \\ \quad s_1s_3 \cdot (-r_1r_2r_3s_2 \cdot x_2 + c_2 - r_1r_3s_2a_1 - r_1a_3) + \\ \quad s_1s_2 \cdot (F - r_2a_1 - r_1r_2s_1s_2s_3a_2) = r_1r_2r_3s_2 \cdot x_3. \end{cases} \tag{32}$$

Substitute for symbol F in all identities (32) its expression from (29). Then, after combining similar terms, as it is easy to verify, we get the following identities:

$$\begin{cases} I.\, -(r_2 + r_1r_3s_1s_2s_3 + r_1r_3s_2 + r_2s_1s_2)a_1 - \\ \quad -(r_1r_2s_3 + r_3s_1s_2 + r_3s_2s_3 + r_1r_2s_1s_2s_3)a_2 - \\ \quad -(r_1s_1s_3 + r_2r_3s_2 + r_2r_3s_1 + r_1)a_3 + c_2 + s_1s_2 \cdot c_3 = 0, \\ II.\, -(r_2s_1s_2 + r_1r_3s_3 + r_1r_3s_2 + r_2)a_1 - \\ \quad -(r_3s_2s_3 + r_1r_2s_1s_2s_3 + r_3 + r_1r_2s_3)a_2 - \\ \quad -(r_2r_3s_1s_2s_3 + r_1s_2s_3 + r_2r_3s_1 + r_1s_1s_3)a_3 + c_3 + s_2s_3 \cdot c_1 = 0, \\ III.\, -(r_1r_3s_1s_2s_3 + r_2s_1s_3 + r_1r_3s_2 + r_2s_1s_2)a_1 - \\ \quad -(r_3 + r_1r_2s_1 + r_3s_2s_3 + r_1r_2s_3)a_2 - \\ \quad -(r_2r_3s_1 + r_1s_1s_3 + r_1 + r_2r_3s_1s_2s_3)a_3 + c_1 + s_1s_3 \cdot c_2 = 0. \end{cases} \tag{33}$$

Let us denote by $-b_1$, $-b_2$, $-b_3$ integer algebraic expressions consisting of all terms of the left sides of the respective identities (33), except those containing certain elements of c_1, c_2, c_3. By implication, the result of all operations implementation in each of these expressions is, respectively, a certain well defined element from a ring K. Therefore, the system of equalities (33) is the following system of linear equations with unknowns c_1, c_2, c_3:

$$\begin{cases} c_2 + s_1s_2 \cdot c_3 &= b_1, \\ c_3 + s_2s_3 \cdot c_1 &= b_2, \\ c_1 + s_1s_3 \cdot c_2 &= b_3 \end{cases} \tag{34}$$

with fixed arbitrary $r_i \in K'$, $a_i \in K$ $(i = 1, 2, 3)$. It is easy to verify that in the ring K of characteristic 3 the system of equations (34) has the following general solution:

$$\left[\begin{array}{lcl} c_1 &=& s_1s_3b_1 - s_2s_3b_2 - b_3, \\ c_2 &=& -b_1 + s_1s_2b_2 - s_1s_3b_3, \\ c_3 &=& -s_1s_2b_1 - b_2 + s_2s_3b_3. \end{array}\right. \tag{35}$$

It is easy to see that all the process of transition from identities (26) to the system of identities (33) is reversible, i.e. (33)$\Leftrightarrow$ (26).

Thus, there really exist such elements $c_1, c_2, c_3 \in K$ which can be defined by formulae (35) and satisfy identities (33) as well as identities (26), with elements c_j $(j = 1, 2, 3)$ from formula (27) being just these elements c_j $(j = 1, 2, 3)$, which can be defined by formulae (35).

By this, we have proved that for loop $Q(A)$, defined on the set $Q = K' \times K$ by formula (18) and with fixed arbitrary sequence $\bar{\bar{a}} = \langle r_i, a_i \rangle_{i=1}^3 \in Q^3$, for its LP-isotope $Q(B)$ there exists a system of substitutions $\tilde{\nu}_{ij}$ $(i, j = 1, 2, 3)$ of the set Q, which can be defined by the formula (27) and satisfy identities (26).

Therefore, by Definition 1, this LP-isotope $Q(B)$ is a IP-loop. So, since LP-isotope $Q(B)$ of 3-loop $Q(A)$ is already considered with fixed arbitrary sequence $\bar{\bar{a}} \in Q^3$, then $Q(B)$ is any LP-isotope of the loop $Q(A)$, being an IP-loop. Then, by Theorem 1, loop $Q(A)$ with operation defined on the set $Q = K' \times K$ by formula (18) is a ternary Moufang loop. This loop is noncommutative and is not a 3-group.

Indeed, the following inequality takes place for operation (18):

$$A(\langle s_1, x_1\rangle, \langle s_2, x_2\rangle, \langle s_3, x_3\rangle) \neq A(\langle s_2, x_2\rangle, \langle s_1, x_1\rangle, \langle s_3, x_3\rangle),$$

i.e. $Q(A)$ is noncommutative. The notion of 3-group for quasigroup $Q(A)$ on the set $Q = K' \times K$ is defined by the following identities:

$$A(A(\langle s_1, x_1\rangle, \langle s_2, x_2\rangle, \langle s_3, x_3\rangle), \langle s_4, x_4\rangle, \langle s_5, x_5\rangle) =$$

$$= A(\langle s_1, x_1\rangle, A(\langle s_2, x_2\rangle, \langle s_3, x_3\rangle, \langle s_4, x_4\rangle), \langle s_5, x_5\rangle) =$$

$$= A(\langle s_1, x_1\rangle, \langle s_2, x_2\rangle, A(\langle s_3, x_3\rangle, \langle s_4, x_4\rangle, \langle s_5, x_5\rangle)).$$

For Moufang 3-loop, defined by formula (18), these identities are not met for all $s_i \in K'$, $x_i \in K$ $(i = 1, 2, 3)$. For example, with $x_2 = x_3 = x_4 = x_5 = 0$, $x_1 \neq 0$,

$s_3 \neq 1$, as it can be verified, not all of these identities are met, i.e. $Q(A)$ is not a 3-group. It can be considered that this Moufang 3-loop is constructed with the help of the initial ring $K = Z_3$.

So, a ternary noncommutative Moufang loop which is not a 3-group is constructed.

4 Example of a ternary commutative Moufang loop different from 3-ary group

In [4, p. 42] a method for the construction of complete ring R_2 of matrices of order n over arbitrary ring R is given. Let us apply this method in the following particular case.

Let Z_2 be a ring of residue classes modulo 2. Operations of addition and multiplication of the ring Z_2 are:

$$\begin{array}{c|cc} + & \bar{0} & \bar{1} \\ \hline \bar{0} & \bar{0} & \bar{1} \\ \bar{1} & \bar{1} & \bar{0} \end{array} \qquad\qquad \begin{array}{c|cc} \cdot & \bar{0} & \bar{1} \\ \hline \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{1} \end{array}$$

Let us consider set M of all possible upper triangular square matrices of order 4 of the following form :

$$x = \begin{pmatrix} \bar{x}_{11} & \bar{0} & \bar{x}_{13} & \bar{x}_{14} \\ \bar{0} & \bar{x}_{11} & \bar{0} & \bar{x}_{24} \\ \bar{0} & \bar{0} & \bar{x}_{11} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{x}_{11} \end{pmatrix}$$

with elements from Z_2, where $\bar{x}_{ii} = \bar{x}_{11}$ for all $i = 1, 2, 3, 4$ and $\bar{x}_{11}, \bar{x}_{13}, \bar{x}_{14}, \bar{x}_{24}$ are any elements from Z_2, the rest of the elements are zeroes $\bar{0}$. When defining by usual way addition and multiplication for them, we get, as it is easy to verify, associative (since Z_2 is associative), but noncommutative ring $M(+, \cdot)$ with identity element.

Zero matrix $0 = 0_4 = \begin{pmatrix} \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} \end{pmatrix}$, consisting of zeroes $\bar{0}$, serves as a zero element of this ring, and identity (unit) matrix $E_4 = \begin{pmatrix} \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$ plays the role of the identity element 1. Since the initial ring Z_2 has characteristic 2, then the ring M, evidently, has also characteristic 2, i.e., $2x = x + x = 0$ for $\forall x \in M$. The subset $M' = \{1, b, c, d\} \subset M$, where $1 = E_4$ is the identity element of the ring M,

$$b = \begin{pmatrix} \bar{1} & \bar{0} & \bar{1} & \bar{1} \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{pmatrix}, c = \begin{pmatrix} \bar{1} & \bar{0} & \bar{0} & \bar{1} \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{pmatrix}, d = \begin{pmatrix} \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$$

($\bar{0}$ is zero element, $\bar{1}$ is identity element of the ring Z_2) relative to matrices multiplication, i.e., $M'(\cdot)$, as it is easy to verify, is an abelian subgroup in $M^*(\cdot)$ (where $M^* = M \setminus \{0\}$), and such that mapping $x \to s \cdot x$ is a substitution of the set M with any $s \in M'$, and $s^2 = 1$ for $\forall s \in M'$. The order of the group M' is equal to 4. Let us define the following ternary operation on the set $Q = M' \times M$ of ordered pairs of the form $\langle s, x\rangle \in Q$.

$$A(\langle s_1, x_1\rangle, \langle s_2, x_2\rangle, \langle s_3, x_3\rangle) = \langle s_1 s_2 s_3, s_2 s_3 \cdot x_1 + s_1 s_3 \cdot x_2 + s_1 s_2 \cdot x_3 + \varphi(s_1, s_2, s_3)\rangle \tag{36}$$

for $\forall s_1, s_2, s_3 \in M', \forall x_1, x_2, x_3 \in M$, where φ is the 3-ary function on M' with the value in M, which is defined by the following formula:

$$\varphi(s_1, s_2, s_3) = \begin{cases} s_1 s_2 s_3 + l, & \text{where } l \text{ is any fixed element from } M' \\ & \text{if the components } s_1, s_2, s_3 \text{ are pairwise different} \\ & \text{(i.e. } s_1 \neq s_2, s_1 \neq s_3, s_2 \neq s_3), \\ 0 & \text{in other cases} \\ & \text{(i.e., if not all of } s_1, s_2, s_3 \text{ are pairwise different).} \end{cases}$$

It is easy to verify, that the 3-groupoid $Q(A)$ with operation (36) is a 3-loop with identity element $\langle 1, 0\rangle$, where 0 is zero element, 1 is identity element of the ring M.

Let us prove that this loop $Q(A)$ is a commutative Moufang loop.

Really, as we have already showed, the LP-isotope of 3-loop $Q(A)$ is defined by formula (20). From the definition of i-translation of the quasigroup it follows that $L_i^{-1}(\bar{\bar{a}})\langle s_i, x_i\rangle$ is an ordered pair of the form $\langle s, x\rangle$ from $Q = M' \times M$, i.e., equality (21) takes place. Then identities (22) are equivalent.

Applying formula (36) to identities (22), the last become as follows:

$$\begin{aligned} 1^\circ.\langle s_1, x_1\rangle &= \langle t_1 r_2 r_3, r_2 r_3 \cdot y_1 + t_1 r_3 \cdot a_2 + t_1 r_2 \cdot a_3 + \varphi(t_1, r_2, r_3)\rangle, \\ 2^\circ.\langle s_2, x_2\rangle &= \langle r_1 t_2 r_3, t_2 r_3 \cdot a_1 + r_1 r_3 \cdot y_2 + r_1 t_2 \cdot a_3 + \varphi(r_1, t_2, r_3)\rangle, \\ 3^\circ.\langle s_3, x_3\rangle &= \langle r_1 r_2 t_3, r_2 t_3 \cdot a_1 + r_1 t_3 \cdot a_2 + r_1 r_2 \cdot y_3 + \varphi(r_1, r_2, t_3)\rangle \end{aligned}$$

for $\forall s_i, r_i, t_i \in M'$ and $\forall a_i, y_i \in M$ $(i = 1, 2, 3)$, where from the following equalities result: $t_1 = r_2 r_3 s_1$, $t_2 = r_1 r_3 s_2$, $t_3 = r_1 r_2 s_3$,

$$y_1 = r_2 r_3 \cdot (x_1 - r_2 s_1 \cdot a_2 - r_3 s_1 \cdot a_3 - \varphi(r_2 r_3 s_1, r_2, r_3)),$$

$$y_2 = r_1 r_3 \cdot (x_2 - r_1 s_2 \cdot a_1 - r_3 s_2 \cdot a_3 - \varphi(r_1, r_1 r_3 s_2, r_3)),$$

$$y_3 = r_1 r_2 \cdot (x_3 - r_1 s_3 \cdot a_1 - r_2 s_3 \cdot a_2 - \varphi(r_1, r_2, r_1 r_2 s_3)).$$

And, since the ring M has characteristic 2, i.e. $-x = x$ for $\forall x \in M$, then the following identities result:

$$\langle t_1, y_1\rangle = \langle r_2 r_3 s_1, r_2 r_3 \cdot (x_1 + r_2 s_1 \cdot a_2 + r_3 s_1 \cdot a_3 + \varphi(r_2 r_3 s_1, r_2, r_3))\rangle,$$

$$\begin{array}{rcl}\langle t_2,y_2\rangle & = & \langle r_1r_3s_2, r_1r_3\cdot(x_2+r_1s_2\cdot a_1+r_3s_2\cdot a_3+\varphi(r_1,r_1r_3s_2,r_3))\rangle, \\ \langle t_3,y_3\rangle & = & \langle r_1r_2s_3, r_1r_2\cdot(x_3+r_1s_3\cdot a_1+r_2s_3\cdot a_3+\varphi(r_1,r_2,r_1r_2s_3))\rangle.\end{array} \quad (37)$$

In view of identities (21)$\wedge$(37) it follows that in the current case identity (20) equivalently transforms in the following way:

$$B(\langle s_1,x_1\rangle,\langle s_2,x_2\rangle,\langle s_3,x_3\rangle)=A(L_1^{-1}(\bar{\bar a})\langle s_1,x_1\rangle,L_2^{-1}(\bar{\bar a})\langle s_2,x_2\rangle,L_3^{-1}(\bar{\bar a})\langle s_3,x_3\rangle)\Longleftrightarrow$$

$$B(\langle s_1,x_1\rangle,\langle s_2,x_2\rangle,\langle s_3,x_3\rangle)=A(\langle r_2r_3s_1,r_2r_3\cdot(x_1+r_2s_1\cdot a_2+r_3s_1\cdot a_3+\varphi(r_2r_3s_1,r_2,r_3))\rangle,$$
$$\langle r_1r_3s_2,r_1r_3\cdot(x_2+r_1s_2\cdot a_1+r_3s_2\cdot a_3+\varphi(r_1,r_1r_3s_2,r_3))\rangle,$$
$$\langle r_1r_2s_3,r_1r_2\cdot(x_3+r_1s_3\cdot a_1+r_2s_3\cdot a_2+\varphi(r_1,r_2,r_1r_2s_3))\rangle).$$

Applying formula (36) to the right side of the last identity, we get the following identity:

$$\begin{array}{rl} B(\langle s_1,x_1\rangle,\langle s_2,x_2\rangle,\langle s_3,x_3\rangle) & = \\ \langle s_1s_2s_3, s_2s_3\cdot(x_1+r_2s_1\cdot a_2+r_3s_1\cdot a_3+\varphi(r_2r_3s_1,r_2,r_3)) & + \\ +s_1s_3\cdot(x_2+r_1s_2\cdot a_1+r_3s_2\cdot a_3+\varphi(r_1,r_1r_3s_2,r_3)) & + \\ +s_1s_2\cdot(x_3+r_1s_3\cdot a_1+r_2s_3\cdot a_2+\varphi(r_1,r_2,r_1r_2s_3)) & + \\ \varphi(r_2r_3s_1,r_1r_3s_2,r_1r_2s_3)\rangle. & \end{array} \quad (38)$$

Thus, if a 3-loop $Q(A)$ is defined on the set $Q=M'\times M$ by formula (36), then its arbitrary LP-isotope $Q(B)$ can be defined by formula (38).

Let us demonstrate that for 3-loop $Q(A)$ defined by formula (36) its LP-isotope $Q(B)$, that can be defined by formula (38), is a IP-loop. To that end, let us consider the same identities (26) relative to the loop $Q(B)$, defined on the set $Q=M'\times M$.

$$B(\{\tilde\nu_{ij}\langle s_j,x_j\rangle\}_{j=1}^{i-1},B(\langle s_j,x_j\rangle_{j=1}^{3}),\{\tilde\nu_{ij}\langle s_j,x_j\rangle\}_{j=i+1}^{3})=\langle s_i,x_i\rangle$$

for $\forall s_i\in M'$, $\forall x_i\in M$ and for every $i=1,2,3$.

Let us define substitutions ν_{ij} $(i,j=1,2,3)$ of the set Q for these identities by the following equalities:

$$\tilde\nu_{ij}\langle s_j,x_j\rangle=\begin{cases}\langle s_j,x_j+c_j\rangle & \text{if } j\neq i,\\ \langle s_j,x_j\rangle & \text{if } j=i\end{cases} \quad (39)$$

for $\forall s_j\in M'$, $\forall x_j\in M$ $(j=1,2,3)$ and for $\forall i=1,2,3$ with fixed arbitrary sequence $\bar{\bar a}=\langle r_i,a_i\rangle\in Q^3$, where $\bar{\bar c}_j$ $(j=1,2,3)$ are some elements of the set M, which are to be determined.

Let us demonstrate that there exist such elements $c_j (j=1,2,3)$ of the set M that, relative to the loop $Q(B)$, substitutions (39) defined by formula (38) meet identities (26).

Let us at first suppose that there already exist such elements c_{ij} $(i,j=1,2,3)$ that the substitutions $\tilde\nu_{ij}$ $(i,j=1,2,3)$ of the set Q, defined by formulae (39), meet identities (26). As we have already demonstrated, identities (26) are equivalent to the system of three identities (28).

$$\begin{cases} I.\ B(B(\langle s_j, x_j\rangle_{j=1}^3), \tilde{\nu}_{12}\langle s_2, x_2\rangle, \tilde{\nu}_{13}\langle s_3, x_3\rangle) = \langle s_1, x_1\rangle, \\ II.\ B(\tilde{\nu}_{21}\langle s_1, x_1\rangle, B(\langle s_j, x_j\rangle_{j=1}^3), \tilde{\nu}_{23}\langle s_3, x_3\rangle) = \langle s_2, x_2\rangle, \\ III.\ B(\tilde{\nu}_{31}\langle s_1, x_1\rangle, \tilde{\nu}_{32}\langle s_2, x_2\rangle, B(\langle s_j, x_j\rangle_{j=1}^3)) = \langle s_3, x_3\rangle. \end{cases}$$

Let us denote the second component of the right side of formula (38) by Φ:

$$\begin{aligned} \Phi = {} & s_2 s_3 \cdot (x_1 + r_2 s_1 \cdot a_2 + r_3 s_1 \cdot a_3 + \varphi(r_2 r_3 s_1, r_2, r_3)) + \\ + {} & s_1 s_3 \cdot (x_2 + r_1 s_2 \cdot a_1 + r_3 s_2 \cdot a_3 + \varphi(r_1, r_1 r_3 s_2, r_3)) + \\ + {} & s_1 s_2 \cdot (x_3 + r_1 s_3 \cdot a_1 + r_2 s_3 \cdot a_2 + \varphi(r_1, r_2, r_1 r_2 s_3)) + \\ + {} & \varphi(r_2 r_3 s_1, r_1 r_3 s_2, r_1 r_2 s_3). \end{aligned} \tag{40}$$

Applying formulae (38) and (40) to the left sides of identities (28), we get that identities (28) are equivalent to the following system of identities:

$$\begin{cases} I.\ B(\langle s_1 s_2 s_3, \Phi\rangle, \langle s_2, x_2 + c_2\rangle, \langle s_3, x_3 + c_3\rangle) = \langle s_1, x_1\rangle, \\ II.\ B(\langle s_1, x_1 + c_1\rangle, B(\langle s_1 s_2 s_3, \Phi\rangle), \langle s_3, x_3 + c_3\rangle) = \langle s_2, x_2\rangle, \\ III.\ B(\langle s_1, x_1 + c_1\rangle, \langle s_2, x_2 + c_2\rangle, \langle s_1 s_2 s_3, \Phi\rangle) = \langle s_3, x_3\rangle. \end{cases} \tag{41}$$

Applying now the same formula (38) to the left sides of identities (41), but relative to new components of the operation B, we get the following equivalent system of three identities:

$$\begin{cases} I.\ \langle s_1 s_2 s_3 \cdot s_2 \cdot s_3, s_2 s_3 \cdot \\ \cdot(\Phi + r_2 s_1 s_2 s_3 \cdot a_2 + r_3 s_1 s_2 s_3 \cdot a_3 + \varphi(r_2 r_3 s_1 s_2 s_3, r_2, r_3)) + \\ + s_1 s_2 s_3 s_3 \cdot (x_2 + c_2 + r_1 s_2 \cdot a_1 + r_3 s_2 \cdot a_3 + \varphi(r_1, r_1 r_3 s_2, r_3)) + \\ + s_1 s_2 s_3 s_2 \cdot (x_3 + c_3 + r_1 s_3 \cdot a_1 + r_2 s_3 \cdot a_2 + \varphi(r_1, r_2, r_1 r_2 s_3)) + \\ + \varphi(r_2 r_3 s_1 s_2 s_3, r_1 r_3 s_2, r_1 r_2 s_3)\rangle = \langle s_1, x_1\rangle, \\ II.\ \langle s_1 \cdot s_1 s_2 s_3 \cdot s_3, s_1 s_2 s_3 s_3 \cdot \\ \cdot(x_1 + c_1 + r_2 s_1 \cdot a_2 + r_3 s_1 \cdot a_3 + \varphi(r_2 r_3 s_1, r_2, r_3)) + \\ + s_1 s_3 \cdot (\Phi + r_1 s_1 s_2 s_3 \cdot a_1 + r_3 s_1 s_2 s_3 \cdot a_3 + \varphi(r_1, r_1 r_3 s_1 s_2 s_3, r_3)) + \\ + s_1 s_1 s_2 s_3 \cdot (x_3 + c_3 + r_1 s_3 \cdot a_1 + r_2 s_3 \cdot a_2 + \varphi(r_1, r_2, r_1 r_2 s_3)) + \\ \varphi(r_2 r_3 s_1, r_1 r_3 s_1 s_2 s_3, r_1 r_2 s_3)\rangle = \langle s_2, x_2\rangle, \\ III.\ \langle s_1 \cdot s_2 \cdot s_1 s_2 s_3, s_2 s_1 s_2 s_3 \cdot \\ \cdot(x_1 + c_1 + r_2 s_1 \cdot a_2 + r_3 s_1 \cdot a_3 + \varphi(r_2 r_3 s_1, r_2, r_3)) + \\ + s_1 s_1 s_2 s_3 \cdot (x_2 + c_2 + r_1 s_2 \cdot a_1 + r_3 s_2 \cdot a_3 + \varphi(r_1, r_1 r_3 s_2, r_3)) + \\ + s_1 s_2 \cdot (\Phi + r_1 s_1 s_2 s_3 \cdot a_1 + r_2 s_1 s_2 s_3 \cdot a_2 + \varphi(r_1, r_2, r_1 r_2 s_1 s_2 s_3)) + \\ + \varphi(r_2 r_3 s_1, r_1 r_3 s_2, r_1 r_2 s_1 s_2 s_3)\rangle = \langle s_3, x_3\rangle. \end{cases} \tag{42}$$

In view of properties of the ring M and its subgroup $M'(\cdot)$ and equality (40), as it is easy to verify, the system of identities (42), after combining similar terms in them, turns into the following system of three identities:

$$\begin{cases} I.\ s_2s_3\cdot[s_2s_3\cdot\varphi(r_2r_3s_1,r_2,r_3)+ & s_1s_3\cdot\varphi(r_1,r_1r_3s_2,r_3)+ \\ +s_1s_2\cdot\varphi(r_1,r_2,r_1r_2s_3)+ & \varphi(r_2r_3s_1,r_1r_3s_2,r_1r_2s_3)]+ \\ +s_2s_3\cdot\varphi(r_2r_3s_1s_2s_3,r_2,r_3)+ & s_1s_2\cdot\varphi(r_1,r_1r_3s_2,r_3)+ \\ +s_1s_3\cdot\varphi(r_1,r_2,r_1r_2s_3)+ & \varphi(r_2r_3s_1s_2s_3,r_1r_3s_2,r_1r_2s_3)+ \\ + & s_1s_2\cdot c_2+s_1s_3\cdot c_3=0, \\ II.\ s_1s_2\cdot\varphi(r_2r_3s_1,r_2,r_3)+ & s_1s_3\cdot[s_2s_3\cdot\varphi(r_2r_3s_1,r_2,r_3)+ \\ +s_1s_3\cdot\varphi(r_1,r_1r_3s_2,r_3)+ & s_1s_2\cdot\varphi(r_1,r_2,r_1r_2s_3)+ \\ +\varphi(r_2r_3s_1,r_1r_3s_2,r_1r_2s_3)]+ & s_1s_3\cdot\varphi(r_1,r_1r_3s_1s_2s_3,r_3)+ \\ +s_2s_3\cdot\varphi(r_1,r_2,r_1r_2s_3)+ & \varphi(r_2r_3s_1,r_1r_3s_1s_2s_3,r_1r_2s_3)+ \\ + & s_1s_2\cdot c_1+s_2s_3\cdot c_3=0, \\ III.\ s_1s_3\cdot\varphi(r_2r_3s_1,r_2,r_3)+ & s_2s_3\cdot\varphi(r_1,r_1r_3s_2,r_3)+ \\ +s_1s_2\cdot[s_2s_3\cdot\varphi(r_2r_3s_1,r_2,r_3)+ & s_1s_3\cdot\varphi(r_1,r_1r_3s_2,r_3)+ \\ +s_1s_2\cdot\varphi(r_1,r_2,r_1r_2s_3)+ & \varphi(r_2r_3s_1,r_1r_3s_2,r_1r_2s_3)]+ \\ +s_1s_2\cdot\varphi(r_1,r_2,r_1r_2s_1s_2s_3)+ & \varphi(r_2r_3s_1,r_1r_3s_2,r_1r_2s_1s_2s_3)+ \\ + & s_1s_3\cdot c_1+s_2s_3\cdot c_3=0. \end{cases} \tag{43}$$

After opening parentheses and in view of properties of the ring M and its subgroup M', the system of identities (43) becomes as follows:

$$\begin{cases} I.\ \varphi(r_2r_3s_1,r_2,r_3)+ & s_2s_3\cdot\varphi(r_2r_3s_1,r_1r_3s_2,r_1r_2s_3)+ \\ +s_2s_3\cdot\varphi(r_2r_3s_1s_2s_3,r_2,r_3)+ & \varphi(r_2r_3s_1s_2s_3,r_1r_3s_2,r_1r_2s_3)+ \\ +s_1s_2\cdot c_2+s_1s_3\cdot c_3= & 0, \\ II.\ \varphi(r_1,r_1r_3s_2,r_3)+ & s_1s_3\cdot\varphi(r_2r_3s_1,r_1r_3s_2,r_1r_2s_3)+ \\ +s_1s_3\cdot\varphi(r_1,r_1r_3s_1s_2s_3,r_3)+ & \varphi(r_2r_3s_1,r_1r_3s_1s_2s_3,r_1r_2s_3)+ \\ s_1s_2\cdot c_1+s_2s_3\cdot c_3= & 0, \\ III.\ \varphi(r_1,r_2,r_1r_2s_3)+ & s_1s_2\cdot\varphi(r_2r_3s_1,r_1r_3s_2,r_1r_2s_3)+ \\ +s_1s_2\cdot\varphi(r_1,r_2,r_1r_2s_1s_2s_3)+ & \varphi(r_2r_3s_1,r_1r_3s_2,r_1r_2s_1s_2s_3)+ \\ +s_1s_3\cdot c_1+s_2s_3\cdot c_3= & 0. \end{cases} \tag{44}$$

In identities (44) we formally truncate all those terms containing function φ for which not all their components are pairwise different, since the value of these terms is equal to 0, and that is why they do not affect the identities implementing. For each of the remaining terms, containing function φ, its components are pairwise different. Therefore, in view of formula (36) and properties of the ring M and its subgroup M', the identities system (44) becomes as follows:

$$\begin{cases} I.\ s_1 + l + s_2s_3 \cdot (s_1s_2s_3 + l) + & s_2s_3 \cdot (s_1s_2s_3 + l) + \\ +s_1 + l + s_1s_2 \cdot c_2 + s_1s_3 \cdot c_3 = & 0, \\ II.\ s_2 + l + s_1s_3 \cdot (s_1s_2s_3 + l) + & s_1s_3 \cdot (s_1s_2s_3 + l) + \\ +s_2 + l + s_1s_2 \cdot c_1 + s_2s_3 \cdot c_3 = & 0, \\ III.\ s_3 + l + s_1s_2 \cdot (s_1s_2s_3 + l) + & s_1s_2 \cdot (s_1s_2s_3 + l) + \\ +s_3 + l + s_1s_3 \cdot c_1 + s_2s_3 \cdot c_2 = & 0 \end{cases} \tag{45}$$

for $\forall s_1, s_2, s_3 \in M'$ and fixed $l \in M'$.

It is easy to see that the system of equalities (45) represents the following system of linear homogeneous equations relative to the unknowns c_1, c_2, c_3:

$$\begin{cases} s_1s_2 \cdot c_2 + s_1s_3 \cdot c_3 & = 0, \\ s_1s_2 \cdot c_1 + s_2s_3 \cdot c_3 & = 0, \\ s_1s_3 \cdot c_1 + s_2s_3 \cdot c_2 & = 0 \end{cases} \Leftrightarrow \begin{cases} s_2c_2 + s_3c_3 = 0, \\ s_1c_1 + s_3c_3 = 0, \\ s_1c_1 + s_2c_2 = 0. \end{cases} \tag{46}$$

In this, it is easy to verify that in this ring M with characteristic 2 the system of equations (46) has the following general solution:

$$c_1 = s_1x, \qquad c_2 = s_2x, \qquad c_3 = s_3x, \tag{47}$$

where x is an arbitrary element from M.

It is easy to see that all the process of transition from identities (26) to the system of equations (46) is reversible, i.e. (46) $\Leftrightarrow$ (26). In this, each nonzero particular solution (47) of the system of equations (46) gives some system of substitutions (39) of the set M which really satisfy identities (26).

Then, according to Definition 1, the loop $Q(B)$ is a JP-loop. Since LP-isotope $Q(B)$ of the loop $Q(A)$ is considered with fixed arbitrary sequence $\bar{\bar{a}} = \langle r_i, a_i \rangle \in Q$, then it follows that $Q(B)$ is any LP-isotope of the loop $Q(A)$, being just JP-loop.

Therefore, by Theorem 1, the loop $Q(A)$, defined on the set $Q = M' \times M$ by formula (36), is a ternary Moufang loop.

It is easy to verify that this loop $Q(A)$ is commutative. On the other hand, it can be easily shown, that for loop $Q(A)$ with operation (36) to be 3-group it is necessary that the following identities to be satisfied:

$$\begin{aligned} s_4s_5 \cdot \varphi(s_1, s_2, s_3) \quad &+ \quad \varphi(s_1s_2s_3, s_4, s_5) = \\ s_1s_5 \cdot \varphi(s_2, s_3, s_4) \quad &+ \quad \varphi(s_1, s_2s_3s_4, s_5) = \\ s_1s_2 \cdot \varphi(s_3, s_4, s_5) \quad &+ \quad \varphi(s_1, s_2, s_3s_4s_5) \end{aligned} \tag{48}$$

for each $s_1, s_2, s_3, s_4, s_5 \in M'$.

But, for the loop $Q(A)$, defined by formula (36), identities (48) are not satisfied for all $s_i \in M'$ $(i = \overline{1,5})$. Thus, according to formula (36), for elements $s_i \in M'$ $(i = \overline{1,5})$ such that s_1, s_2, s_3 are pairwise different and $s_1 \neq 1, s_2 \neq 1, s_3 \neq 1$, but $s_4 = s_5 = 1$, identities (48) reduce to the following identities:

$$\varphi(s_1, s_2, s_3) = s_1 \cdot \varphi(s_2, s_3, 1) + \varphi(s_1, s_2s_3, 1) = \varphi(s_1, s_2, s_3).$$

Since in the subgroup $M' \subset M$ of order 4 product of any two nonunitary elements is equal to the third nonunitary element of this subgroup, then in this case, $s_2s_3 = s_1$. Therefore, the previous identities reduce to the identity:

$$\varphi(s_1, s_2, s_3) = s_1 \cdot \varphi(s_2, s_3, 1)$$

or

$$s_1s_2s_3 + l = s_1 \cdot (s_2s_3 + l),$$

wherefrom the following identity results:

$$l = s_1 \cdot l$$

for $\forall s_1 \in M' \backslash 1$ and with the fixed $l \in M'$.

The last identity evidently is not satisfied for all $s_1 \in M'$. Thus, for 3-loop $Q(A)$, defined by formula (36), identity (48), generally speaking, is not satisfied, i.e., 3-loop $Q(A)$ is not a 3-group. So, ternary commutative Moufang loop, different from 3-group, is constructed.

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References

[1] Belousov V. D. *Foundations of the theory of quasigroups and loops.* Nauka, Moscow, 1967 (in Russian).

[2] Belousov V. D. *n-ary quasigroups.* Ştiinţa, Chisinau, 1972 (in Russian).

[3] Kurosh A. G. *General algebra. Lectures during academic year 1969–1970.* Nauka, Moscow, 1974 (in Russian).

[4] Kurosh A. G. *Lectures on general algebra.* Nauka, Moscow, 1973 (in Russian).

[5] Bertram Wolfgang, Kinyon Michael. *Torsors and ternary Moufang loops arising in projective geometry.* In: Makhlouf A., Paal E., Silvestrov S., Stolin A. (eds), Algebra, Geometry and Mathematical Physics. Springer Proceedings in Mathematics & Statistics, vol 85. Springer, Berlin, Heidelberg, p. 343–360.

[6] Bertram Wolfgang, Kinyon Michael *Torsors and ternary Moufang loops arising in projective geometry.* arXiv:1206.2222, June 2012.

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