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Degree of favoring in apportionments

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Abstract

To quantitatively estimate the degree of favoring the beneficiaries in proportional apportionments of entities of the same kind (seats, PCs, etc.), five quantitative criteria were defined. By computer simulation, the degree of favoring the large or small beneficiaries by 6 apportionment methods is identified. Thus, favoring large beneficiaries by the d'Hondt method can overpass 10.7-12.1 entities (entities in excess) and that of small beneficiaries by the Huntington-Hill method – 2.7-11.0 entities, and by the Adapted Sainte-Laguë method – 1.7-9.7 entities. The Huntington-Hill method favors small beneficiaries up to 5.70 times stronger than the Adapted Sainte-Laguë one does. Also, the d'Hondt method favors beneficiaries (the large ones) much stronger than the Adapted Sainte-Laguë one does (the small ones) – for very many cases the respective ratio exceeds 10 times.

Keywords: apportionment method, comparative analysis, computer simulation, criteria, favoring large beneficiaries, favoring small beneficiaries.

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1 Introduction

Often it is necessary to distribute a given number M of discrete entities of the same kind (a-entities) among n beneficiaries (parties, schools, hospitals, etc.) in proportion to a numerical characteristic assigned to each of them $V_i, i = \overline{1,n}$. For example, M may be the total number of seats in the elective body, n – the total number of parties and V – the total number of voters (deciders). This is known as proportional apportionment (APP) problem [1-3]. The integer character of

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this problem usually causes a certain disproportion of the apportionment $\{x_i, i = \overline{1,n}\}$ [1, 4 - 6], some beneficiaries being favored at the expense of the others (notations $\{c_i, i = \overline{1,n}\}$ and $\{c_1, c_2, ..., c_n\}$ are used equivalently to specify the group of elements $c_1, c_2, ..., c_n$). Such favoring leads to the increase of disproportionality of the apportionment. Therefore, reducing the favoring in question is one of the basic requirements (free of bias condition [1, 3]) when the APP method is chosen to be applied under concrete situations.

As it is well known, the d'Hondt method [7] favors large beneficiaries (with larger V_i value) [1, 4, 8, 9], and Huntington-Hill method [10] favors the small ones (with smaller V_i value) [1, 8]. But which of the two favors beneficiaries to a larger extent? Preferences, in this sense, between methods, can help. For example, in [8], five divisor APP methods are placed "in the order as they are known to favor larger parties over smaller parties: Adams, Dean, Hill, Webster, and Jefferson" (in the increasing order of favoring). However, the best way is to estimate this property quantitatively – the degree of favoring the beneficiaries. Three such approaches are proposed in [12, 13]. These and other approaches, including new criteria for estimating the degree of favoring the large or, on the contrary, the small beneficiaries by APP methods, are explored in this paper by computer simulation.

2 Approaches of favoring in apportionments

The essence of favoring the beneficiaries in apportionments is described in different papers, including [1, 4, 8, 11]. Four such approaches are described below in this section.

Approach 1. Let's consider a-entities as seats, deciders as population and beneficiaries as states. Then, according to [12], "An apportionment that gives x_1 and x_2 seats to states having populations $V_1 > V_2 > 0$ favors the larger state over the smaller state if $x_1/V_1 > x_2/V_2$, and favors the smaller state over the larger state if $x_1/V_1 < x_2/V_2$". To compare APP methods with reference to favoring the states, the bias ratio $\delta/(\gamma + \delta)$ as criterion is proposed; here δ is the number of pairs of states in which the small state is favored and γ is the number of pairs of states in which the large state is favored. Using this criteria, the results obtained in [12] for the 19 United States Congressional apportionments of seats in the period of 1790-1970 years are (for five APP methods): J.O.Adams – 0.780; J.Dean – 0.583; E.V.Huntington – 0.562; D.Webster – 0.518 and T.Jefferson – 0.199. APP methods are listed in the decreasing order of the bias ratio of favoring the small states, that is in the increasing order of favoring the large states.

Approach 2 [12]. Let g be a number satisfying $x_i > g$ for at least $\lfloor n/2 \rfloor$ states and $x_i < g$ for at least $\lfloor n/2 \rfloor$ states. Define the set of large states as $L = \{i, x_i > g\}$ and the set of small states as $S = \{i, x_i < g\}$; a state with g seats belongs to both L and S. Then, in an apportionment, the large states are favored if $\Sigma_L x_i / \Sigma_L V_i > \Sigma_S x_i / \Sigma_S V_i$ and the small states are favored if $\Sigma_L x_i / \Sigma_L V_i < \Sigma_S x_i / \Sigma_S V_i$. Applying this criterion to the mentioned above 19 apportionments of seats, in [12], the following results are obtained (number of times small states were favored for each of the five APP methods): J.O.Adams – 19; J.Dean – 14; E.V.Huntington – 13; D.Webster – 9, and T.Jefferson – 0. This order of APP methods coincides with the one obtained in Approach 1.

Approach 3. The same order for the five APP methods, but applying an analytical not experimental approach, is obtained in [8, 11]. The type of ordering used is called majorization (majorization ordering). "It has the advantage of providing a complete characterization, and has its roots in studies of equality and inequality" [8, p. 886] and its formal properties [15]. Majorization provides an ordering between two vectors **m** $= (m_1, \ldots, m_l)$ and **m'** $= (m'_1, \ldots, m'_l)$, with ordered elements $m_1 \geq \ldots, \geq m_l$ and $m'_1 \geq ... \geq m'_l$, and with an identical component sum $m_1 + \ldots, + m_l = m'_1 + ... + m'_l = M$. The ordering states that all partial sums of the k largest components in **m** are dominated by the sum of the k largest components in **m'**: $m_1 \leq m'_1$, $m_1 + m_2 \leq m'_1 + m'_2, \ldots, m_1 + ..., + m_l \leq m'_1 + ... + m'_l$.

To compare APP methods by majorization ordering, the notion of majorization from apportionment vectors to apportionment methods is extended and the signpost sequences that determine the methods are used. So [1], a divisor method of apportionment is defined through numbers $s(k)$ in the interval $[k; k+1]$ such that the sequence

$s(0), s(1), ...$ is strictly increasing; here a number $s(k)$ is a "signpost" ("dividing point") splitting the interval $[k; k+1]$ into a left part, where numbers are rounded down to k, and a right part, where numbers are rounded up to $k+1$. For $s(k)$ itself, there is the option to round down to k or to round up to $k+1$, thus possibly generating multiplicities. The numbers rounded this way are the quotients of the weights (V_i) and a divisor (d), $V_1/d, \ldots, V_l/d$, for some choice of divisor $d > 0$ common to all weights. If party i gets m_i seats, then necessarily $s(m_i–1) \leq V_i/d \leq s(m_i)$. The divisor d is adjusted so that the sum of all seats becomes equal to M.

Taking into account the introduced notions, in [8, 11] it is proved that divisor method A is majorized by divisor method A', in sense of favoring, if and only if the ratio $s(k)/s'(k)$ is strictly increasing in k, where $s(k)$ and $s'(k)$ are the signposts for methods A and A', respectively. Finally, it is proved that the divisor method with power-mean rounding of order p is majorized by the divisor method with power-mean rounding of order p' if and only if $p \leq p'$. This statement puts the mentioned above five divisor methods into the majorization ordering that coincides with the one obtained in Approaches 1 and 2.

Approach 4. A more strong, than the three described above approaches, is the one proposed in [1]: an apportionment method favors large parties if $\Sigma_L x_i/\Sigma_L V_i > \Sigma_S x_i/\Sigma_S V_i$ and it favors small parties if $\Sigma_L x_i/\Sigma_L V_i < \Sigma_S x_i/\Sigma_S V_i$, where L and S are subsets of $1, 2, \ldots, n$ such that $x_i > x_j$ whenever $i \in L$ and $j \in S$ [3].

At the same time, as mentioned in [16], there are no known such methods that would be used in practice; one and the same APP method in some apportionments can favor, predominantly, large beneficiaries, and in other apportionments, predominantly, small beneficiaries. This is why, as noted in [16], this approach can be used to identify the "total favoring" or "full favoring" of large or small beneficiaries in particular apportionments. It is easy to observe that "full favoring" are particular cases of "favoring" the beneficiaries – large (predominantly) or small (predominantly).

Resuming, Approaches 1-4 can place APP methods in the decreasing order of favoring the small or, on the contrary, the large beneficiaries. Moreover, Approaches 1 and 2 permit to characterize quantitatively, to some extent, APP methods: Approach 1 – the relative frequency (probability, on the infinite number of apportionments) of favoring by pairs large or small beneficiaries; Approach 2 – the relative frequency (probability, on the infinite number of apportionments) of favoring the subset of large or the subset of small beneficiaries. In Section 3, there are discussed other informative approaches regarding the degree of favoring the large or the small beneficiaries. First, a systemized vision of favoring the beneficiaries in apportionments is done.

3 Criteria for estimating the degree of favoring

Based on [13, 15], one can distinguish four notions of favoring in apportionments:

a) favoring a decider (voter, etc.) in an apportionment;

b) favoring a beneficiary in an apportionment;

c) favoring large or small beneficiaries in an apportionment;

d) favoring large or small beneficiaries overall by an apportionment method.

Also, as mentioned in [13], each of the specified above four notions can be characterized by:

A) identifying the fact of favoring the deciders or beneficiaries in apportionments;

B) quantitatively estimating the favoring of deciders or beneficiaries in apportionments.

Of course, all quantitative criteria, along with the respective quantitative assessments (aspect B), can be used also to identify the fact of favoring in apportionments (aspect A) [13]. Combining issues A and B with notions (a)-(d), further in this paper the following aspects of favoring in apportionments will be distinguished: Aa, Ab, Ac and Ad and, respectively, Ba, Bb, Bc and Bd.

With refer to proportional apportionments, it is considered that a beneficiary i is favored if a larger number x_i of a-entities is distributed to him than would be due according to the V_i value, that is $x_i > MV_i/V = D_i$, where $M = x_1+x_2+\ldots+x_n$ and $V = V_1+V_2+\ldots+V_n$. In [15], D_i is defined as the free of bias part (rights, influence power, etc.) of beneficiary i in the apportionment; also, $r = M/V$ is defined as the free of bias part (rights, etc.) of a decider in the apportionment, if to consider that V is the total number of deciders. Of course, the lack of favoring is possible only if the equalities $\lfloor MV_i/V \rfloor = MV_i/V, i = \overline{1,n}$ take place; here $\lfloor z \rfloor$ means the integer part of the real number z. In practice, such equalities rarely occur and that is why some beneficiaries are favored and others, respectively, are disfavored.

Definition 1. [15] *In an apportionment, a beneficiary i is favored, if it gets an excess number of a-entities ($\Delta D_i = x_i – D_i > 0$), is disfavored, if it obtains a deficit number of a-entities ($\Delta D_i < 0$), and is neutral (neither favored nor disfavored), if it gets a number of a-entities equal to the expected one ($\Delta D_i = 0$).*

So, the essense of aspect Ab is the following:

1) for the favored beneficiary i, occurs $x_i > a_i$, where $a_i = \lfloor D_i \rfloor$;

2) for the disfavored beneficiary i, occurs $x_i \leq a_i$ at $D_i > a_i$;

3) for the neutral beneficiary i, occurs $x_i = a_i$ at $D_i = a_i$.

Aspect Bb, the degree of favoring the beneficiary i is characterized by the number of a-entities in excess in the apportionment: $\Delta D_i = x_i – D_i$; of course, if $\Delta D_i < 0$, then the beneficiary i is disfavored because it has a deficit of a-entities. Here, it is useful to mention that because of $D_1 + D_2 + \ldots + D_n = M$ and $x_1 + x_2 + \ldots + x_n = M$, if some beneficiaries are favored, the other ones are mandatory disfavored at the same summary extent.

Now, let r_i be the power of influence of a decider that supported the beneficiary i in the apportionment. According to [13], one has $r_i = x_i/Vi, i = \overline{1,n}$.

Definition 2. *(based on [15]) In an apportionment, a decider that supported the beneficiary i is favored if it gets an excess value of influence power ($\Delta r_i = r_i - r > 0$); is disfavored if it obtains a deficit value of influence power ($\Delta r_i < 0$); and is neutral (neither favored nor disfavored) if it gets a value of influence power equal to the expected one ($\Delta r_i = 0$).*

So, the essence of aspect Aa, regarding a decider that supported the beneficiary i, is the following:

1) for the favored decider, occurs $r_i > r$;

2) for the disfavored decider, occurs $r_i < r$;

3) for the neutral decider, occurs $r_i = r$.

Aspect Ba, the degree of favoring of a decider that supported the beneficiary i is characterized by the value of influence power in excess in the apportionment, $\Delta r_i = r_i - r$; of course, if $\Delta r_i < 0$, then the decider is disfavored because it has a deficit of influence power.

Statement 1. *In apportionments, favoring the deciders absolutely correlates with favoring the beneficiaries supported by them.*

Indeed, one has $D_i = MV_i/V$ and $r = M/V$. So, $r = D_i/V_i$. At the same time, $\Delta r_i = r_i - r = x_i/V_i - D_i/V_i = (x_i - D_i)/V_i = \Delta D_i/V_i, i = \overline{1,n}$. ♦

According to Statement 1, if in an apportionment, a beneficiary is favored, then all deciders that supported this beneficiary are favored as well. This is why, above in this section, both these cases (favoring a beneficiary and favoring a decider that supported him) are referred to together as aspect B. However, sometimes it is useful to characterize quantitatively favoring the deciders apart from the characterization of favoring the respective beneficiary.

Statement 2. *The discrepancy of favoring the deciders that supported different beneficiaries (i and $k \neq i$) in apportionments, measured as the difference $\Delta r_i - \Delta r_k$, is equivalent to the one measured as the difference $r_i - r_k$, that is $\Delta r_i - \Delta r_k = r_i - r_k, k \neq i, i = \overline{1,n}$.*

Indeed, one has $\Delta r_i - \Delta r_k = r_i - r - (r_k - r) = r_i - r_k, (i,k) = \overline{1,n}, k \neq i$. ♦

Consequence 1. *Differences $\Delta r_i - \Delta r_k$ and $r_i - r_k$, which characterize the discrepancy of favoring the deciders that supported different beneficiaries (i and $k \neq i$) in apportionments, are interchangeable, but the last one is simpler.*

Thus, in apportionments, the degree of favoring a beneficiary i (aspect Bb) is characterized by the parameter $\Delta D_i (i = \overline{1,n})$, measured in a-entities (aE), and the degree of favoring a decider which supported the beneficiary i (aspect Ba) is characterized by the parameter $\Delta r_i (i = \overline{1,n})$, measured in a-entities/decider (aE/DM). It remains to define criteria for the degree of favoring large or small beneficiaries in an apportionment (aspect Bc), overall by an apportionment method (aspect Bd) and, optionally, of aspects Ac and Ad.

A criterion (F_{a1}) for estimating the degree of favoring the large or small beneficiaries in proportional apportionments (aspect Bc) is proposed in [13]. Further, without diminishing the universality of the approach, it is considered that the n beneficiaries are ordered in the non-ascending order of $V_i, i = \overline{1,n}$ values, that is $V_1 \geq V_2 \geq V_3 \geq \ldots \geq V_n$. Based on these relations, in [12] the L and S subsets of large and, respectively, small beneficiaries were defined as follows: $L = \{1, 2, \ldots, \lfloor n/2 \rfloor\}$ and $S = \{\lceil n/2 \rceil + 1, \lceil n/2 \rceil + 2, \ldots, n\}$, where $x_i \geq x_j$ whenever $i \in L$ and $j \in S$. Here, it should be noted that in proportional apportionments, if $V_i > V_k$, then $x_i \geq x_k$.

Definition 3. [13] *An apportionment favors large beneficiaries if the summary number of a-entities in excess obtained by large beneficiaries (L) is greater than that obtained by small beneficiaries (S), and vice versa; that is, it favors large beneficiaries if $F_{a1} > 0$, it favors the small ones if $F_{a1} < 0$, and it is neutral if $F_{a1} = 0$, where*

$$F_{a1} = \sum_{i \in L} \Delta D_i - \sum_{i \in S} \Delta D_i. \tag{1}$$

Criterion F_{a1} ensures both, the identification of the fact of favoring (aspect Ac) and the estimation of the degree of favoring the large or the small beneficiaries in an apportionment, measured in a-entities (aspect Bc). Similarly, for APP methods, as a quantitative criterion in [13], the average $\overline{F_{a1}}$ of F_{a1} on the infinity of apportionments is proposed.

Definition 4. [13] *An apportionment method favors large beneficiaries if the average summary number of a-entities in excess obtained by large beneficiaries (L) is greater than that obtained by small beneficiaries (S), and vice versa; that is, it favors large beneficiaries if* $\overline{F_{a1}} > 0$*, it favors the small ones if* $\overline{F_{a1}} < 0$*, and it is neutral if* $\overline{F_{a1}} = 0$*, where* $\overline{F_{a1}}$ *is the average of* F_{a1} *on the infinity of apportionments. So,*

$$\overline{F_{a1}} = \lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K}\Big(\sum_{i\in L}\Delta D_{ik} - \sum_{i\in S}\Delta D_{ik}\Big) = \sum_{i\in l}\overline{\Delta D_i} - \sum_{i\in S}\overline{\Delta D_i}, \quad (2)$$

where $\overline{\Delta D_i}$ *is the average of* ΔD_i *on* $K \to \infty$ *apportionments.*

As criterion F_{a1}, the $\overline{F_{a1}}$ one ensures the identification of the fact of favoring (aspect Ad) and the quantitative estimation of the degree of favoring the large or the small beneficiaries by an APP method, measured in a-entities (aspect Bd). Also, from Definition 4 one can conclude that if $\overline{F_{a1}} \neq 0$, then the respective APP method is favoring the beneficiaries (the large ones in case of $\overline{F_{a1}} > 0$ or the small ones in case of $\overline{F_{a1}} < 0$).

Further, the notation $\overline{Y}$ of the average of parameter Y values on the infinity of apportionments will be used.

In addition to criteria $\Delta D_i, \Delta r_i, F_{a1}$ and $\overline{F}_{a1}$ already defined, such quantitative criteria for assessing the favoring of beneficiaries by APP methods may be useful in various situations, especially in research, as:

- The average relative discrepancy between the degree of favoring of an average (conventional) large beneficiary decider and that of an average (conventional) small beneficiary decider $\overline{F_{r1}}$ (%decider-power), where $F_{r1} = 100(r_L - r_S)/r$, $r_L = \sum_{i\in L} x_i / \sum_{i\in L} V_i$, $r_S = \sum_{i\in S} x_i / \sum_{i\in S} V_i$. Here r_L and r_S are the influence power of an average (conventional) large and, respectively, of an average (conventional) small beneficiary decider;

- The average largest absolute discrepancy of the degree of favoring between two beneficiaries $\overline{F_{a0}} = \sum_{i=1}^{n-1}(\overline{\Delta D_i} - \overline{\Delta D_{i+1}}) = \overline{\Delta D_i} - \overline{\Delta D_n}$, a-entities;

- The largest discrepancy of the probability of favoring between two beneficiaries $F_p = \sum_{i=1}^{n-1}(F_{pi} - F_{p,i+1}) = F_{p1} - F_{pn}$. Here F_{pi} is the probability of favoring the beneficiary i in apportionments, that is $F_{pi} = \lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K} C_{ik}$, where $C_{ik} = 1$, if $x_i > D_i$, and $C_{ik} = 0$, otherwise;

- The average largest relative discrepancy of the degree of favoring between two deciders which supported different beneficiaries $\overline{F_{r0}}$ (%decider-power - %DP), where $F_{r0} = 100\sum_{i=1}^{n-1}(r_i - r_{i+1}) = 100(r_1 - r_n)/r$.

Criterion $\overline{F_{a1}}$ allows the absolute evaluation and that of $\overline{F_{r1}}$ – the relative evaluation on r of the degree of favoring the large (L) or the small (S) beneficiaries by the APP method. The next two criteria, $\overline{F_{a0}}$ and F_p, characterize the discrepancy of favoring between the largest ($i = 1$) and the smallest ($i = n$) beneficiaries in the apportionment. Finally, criterion $\overline{F_{r0}}$ characterizes the discrepancy of favoring between a decider which supported the largest ($i = 1$) beneficiary and a decider which supported the smallest ($i = n$) beneficiary; it correlates with $\overline{F_{a0}}$, but is simpler. Anyway, the main criterion, when determining the degree of favoring the beneficiaries by APP methods, is the $\overline{F_{a1}}$ one.

According to Definitions 3 and 4, there is a clear distinction between the degree of favoring the beneficiaries in an apportionment and the degree of favoring the beneficiaries overall by an APP method. For specific apportionments, an APP method may favor particular beneficiaries, both large and small, but overall, on the infinite number of apportionments, be neutral. Namely, the degree of favoring the beneficiaries by an APP method overall, on the infinite number of apportionments, will be investigated thereafter in this paper.

4 Overview of computer simulation of favoring

In total, 6 APP methods are investigated, namely the Hamilton (Hare) - H, d'Hondt (Jefferson) - d'H, Huntington-Hill (HH), Adapted Sainte-Laguë (ASL), Variable linear divisor (VLD) and Quota dependent linear divisor (QDLD) ones. All these methods are described, for example, in [17].

In order to determine the values of quantitative criteria $\overline{F_{a0}}$, $\overline{F_{r0}}$, $\overline{F_{a1}}$, $\overline{F_{r1}}$ and $\overline{F_p}$, computer simulation using SIMAP application was performed. Initial data used in calculations are: $M = 6$, 11, 21, 51, 101, 201, 501; $n = 2$, 3, 4, 5, 7, 10, 15, 20, 30, 50; $n \leq M$–1; $V = 10^8$; uniform distribution of the values $V_i, i = \overline{1, n}$; sample size 10^6. So, we have 58 variants of values for the pair $\{M, n\}$: 4 + 6 + 8 + 10 = 58. The use of small values of M is useful, for example, when determining the M members of a parliamentary committee basing on the number of deputies (V_i) of each of the n parties in the Parliament. Sometimes, from the specified above 58 variants, only 50 are used – variants for which $n \leq M/2$. This is because cases in which $M/2 < n < M$ are rarely encountered in practice, but the apportionment disproportion when applying the Huntington-Hill and adapted Sainte-Laguë methods in such cases increases considerably. Some of the obtained results are described below.

5 Preferenced APP methods by non-favoring the beneficiaries

Some results of calculations for the average value (on 50 variants of the pair of sizes for M and n values) of criteria $\overline{F_{a0}}$, $\overline{F_{a1}}$, $\overline{F_{r1}}$ and $\overline{F_p}$ at $n \leq M/2$ are systemized in Table 1. In more detailes, for the pair $\{M, n\}$ at $n \leq M/2$ one has 50 variants of values described in Section 4. For each such pair of values, 1 mil variants of values for $\{V_i, i = \overline{1, n}\}$ sizes were generated randomly, at uniform distribution, thus being obtained 1 mil variants of the APP problem initial data; for each such variant, calculations were done, and after that, there were obtained the average values of the explored parameters. Finally, there were also calculated the average values of the explored criteria on 50 variants of pair $\{M, n\}$ values. So, for each of the 6 explored methods, calculi were done on $50 \times 1000000 = 50$ mil apportionments.

Table 1. Results for $\overline{F_{a0}}, \overline{F_{a1}}$, $\overline{F_p}$ and $\overline{F_{r1}}$ at $n \leq M/2$

Criterion	APP method and the average value of criterion on 50 variants of the pair $\{M, n\}$ sizes					
$\overline{F_{a1}}$, a-entities	VLD $\succ$	H $\succ$	ASL$\succ$	HH $\succ$	QDLD $\succ$	d'H $\succ$
	0.02014	0.04040	-0.71040	-0.92720	-2.18101	2.50428
$\overline{F_{r1}}$, %DP	VLD $\succ$	H $\succ$	ASL$\succ$	HH $\succ$	QDLD $\succ$	d'H $\succ$
	0.23809	0.34620	-4.08271	-4.99704	-5.42261	8.99951
$\overline{F_{a0}}$, a-entities	VLD $\succ$	H $\succ$	ASL$\succ$	HH $\succ$	QDLD $\succ$	d'H $\succ$
	0.03590	0.03882	-0.26795	-0.31000	-0.42923	0.66386
$100F_p$, %	VLD $\succ$	H $\succ$	ASL$\succ$	HH $\succ$	QDLD $\succ$	d'H $\succ$
	9,840	10,125	-20,290	-24,420	-36,681	68,644

In Table 1, the positive values of used criteria correspond to cases of favoring large beneficiaries, and the negative ones – to cases of favoring small beneficiaries. The values of criteria for the VLD, H and d'H methods are positive, and those for the ASL, HH and QDLD (at $n > 3$) methods are negative. Of course, there is no doubt about the Hamilton method neutrality with refer to favoring the beneficiaries overall, on the infinity of apportionments (see, for example, [17]). Also, the average value of criterion $\overline{F_{a1}}$ for Hamilton method is of 0.04040 a-entities, that is 17.6 times smaller than that for Adapted Sainte-Laguë method. So, taking into account the limited precision of computer simulation, according to Table 1, one can conclude that:

1) VLD and H (as is well-known) methods are neutral in favoring the beneficiaries;
2) ASL, HH (as is well-known) and QDLD (at $n > 3$) methods favor small beneficiaries;
3) d'H method favors large beneficiaries (as is well-known).

It should be mentioned that in Table 1 the relation A $\succ$ B of method A preference to method B is done by the absolute value of used criteria. Also, despite the limited accuracy, the results of calculations are obtained for the same initial data sets and can therefore be used in the comparative analysis for all methods.

From Table 1, one can see that preferences of APP methods by criteria $\overline{F_{a1}}, \overline{F_{r1}}$ and F_p coincide. So, all criteria $\overline{F_{r1}}, \overline{F_{a0}}$ and F_p can be used to identify the fact of favoring by APP methods, but the last two are simpler than the $\overline{F_{a1}}$ and $\overline{F_{r1}}$ ones. Also, one can say that the preferences of examined 6 APP methods in non-favoring of beneficiaries are the following: VLD $\succ$ H $\succ$ ASL $\succ$ HH $\succ$ QDLD $\succ$ d'H.

Obtained average values of the explored criteria on the 50 variants of the pair $\{M, n\}$ values allow, to some extent, the overall determination of the APP methods preferences regarding the favoring of beneficiaries. At the same time, additional information can be obtained using similar calculations for each of the 58 variants of the pair $\{M, n\}$ values.

Overall, the degree of favoring of beneficiaries by APP methods is determined by criterion $\overline{F_{a1}}$ or the $\overline{F_{r1}}$ one. The more specific criteria $\overline{F_{a0}}$, $\overline{F_{r0}}$, and F_p can also be used for this purpose. Selective results of calculations according to criteria $\overline{F_{a1}}$, $\overline{F_{r1}}$, $\overline{F_{a0}}$, and $\overline{F_{r0}}$ for the d'Hondt, Adapted Sainte-Laguë, Huntington-Hill, and QDLD methods are described in Section 6.

6 Degree of favoring by some APP methods

The graphs of criteria $\overline{F_{a0}}, \overline{F_{a1}}$, and $\overline{F_{r1}}$ dependence on M and n for **d'Hondt method** are shown in Figures 1-2a. Characterizing the largest absolute discrepancy of the degree of favoring between two beneficiaries, the criterion $\overline{F_{a0}}$ (Figure 1) most easily identifies the fact of favoring the large or the small beneficiaries in case of separate pairs of values of sizes M and n.

From Figure 1a, it can be seen that out of the 58 variants of the pair $\{M, n\}$ values, the largest average absolute discrepancy $\overline{F_{a0}}$(d'H) between the largest ($i = 1$) and the smallest ($i = n$) beneficiary is obtained for the pair $\{M = 501, n = 20\}$, this being equal to approx. 0.9 a-entities. For $\{6 \leq M \leq 501,\ M \geq 2n\}$, this discrepancy is in the range of $0.32 \div 0.39$ aE at $n = 2$, of $0.51 \div 0.56$ aE at $n = 3$, of $0.62 \div 0.65$ aE at $n = 4$, of $0.71 \div 0.84$ aE at $n = 10$, of $0.69 \div 0.89$ aE at $n = 20$, and of $0.55 \div 0.79$ aE at $n = 50$, being considerable.

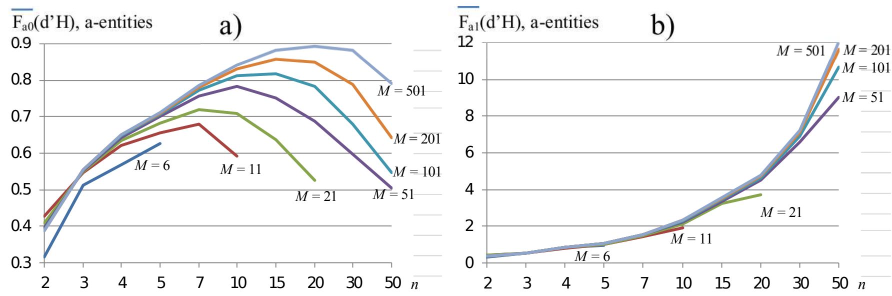

Figure 1. Criteria $\overline{F_{a0}}$ and $\overline{F_{a1}}$ dependences to M and n for d'H method.

Figure 1b shows that the value $\overline{F_{a1}}$(d'H) is increasing to n and slightly increasing to M, especially at $M \geq 2n$. For $\{6 \leq M \leq 501, 2 \leq n \leq 50,\ n < M\}$, the $\overline{F_{a1}}$(d'H) value is between $0.32 \div 0.41$ aE (at $n = 2$), and $9.0 \div 12.1$ aE (at $n = 50$), being considerable at relatively high values of n. Under the same conditions, but $n \leq M/2$, the $\overline{F_{a1}}$(d'H) value at $n = 50$ is in the range of $10.7 \div 12.1$ aE.

Criterion $\overline{F_{r1}}$ compares the average influence of a decider belonging to the group of large beneficiaries ($i = 1, 2, \ldots, \lfloor n/2 \rfloor$) with that of a decider belonging to the group of small beneficiaries ($i = \lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n$). The discrepancy $\overline{F_{r1}}$(d'H) is decreasing with respect to M and is increasing with respect to n (Figure 2a). At high values of n and low values of M, this discrepancy is considerable. Thus, for initial data stated in Section 4, they belong to the range of $0.43 \div 35.32$ %decider-power at $n = 4$, of $0.73 \div 53.00$ %DP at $n = 5$, of $1.20 \div 44.77$ %DP at $n = 10$, and of $6.36 \div 46.67$ %DP at $n = 50$. If $n \leq M/2$, the discrepancy $\overline{F_{r1}}$(d'H) is within the range of $0.43 \div 18.50$ %DP at $n = 4$, of $0.73 \div 30.88$ %DP at $n = 5$, of $1.20 \div 26.28$ %DP at $n = 10$, and of $6.36 \div 27.90$ %DP at $n = 50$.

The graphs of criteria $\overline{F_{a0}}, \overline{F_{a1}}$, and $\overline{F_{r1}}$ values dependence on M and n, using the Huntington-Hill method, are shown in Figures 2b-3.

Figure 2b shows that discrepancy $|\overline{F_{r1}}(\mathrm{HH})|$ is decreasing on M and is increasing on n. For $\{6 \leq M \leq 501,\ 2 \leq n \leq 50,\ n \leq M/2\}$, the $|\overline{F_{r1}}(\mathrm{HH})|$ values belong to the range of $0.0014 \div 10.63$ %DP at $n = 2$, of $0.0092 \div 33.36$ %DP at $n = 3$, of $0.010 \div 12.01$ %DP at $n = 4$, of

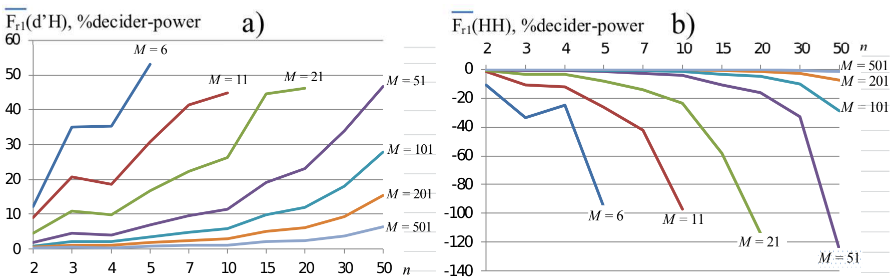

Figure 2. Criterion $\overline{F_{r1}}$ dependence to M and n for d'H and HH methods.

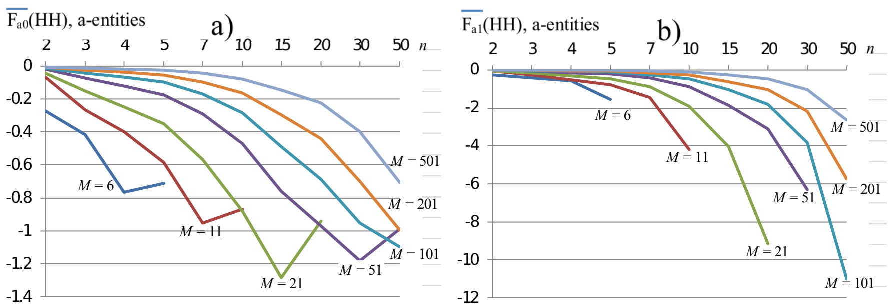

Figure 3. Criteria $\overline{F_{a0}}$ and $\overline{F_{a1}}$ dependences to $\{M, n\}$ for HH method.

$0.065 \div 23.33$ %DP at $n = 10$, and of $1.40 \div 28.80$ %DP at $n = 50$.

According to Figure 3a, out of the 58 variants of the pair $\{M,\, n\}$ values, the largest average absolute discrepancy between the largest ($i = 1$) and the smallest ($i = n$) beneficiary, $\overline{F_{a0}}$(HH), is obtained in case of $\{M = 21,\ n = 15\}$ and is equal to approx. -1.28 a-entities. The discrepancy $|\overline{F_{a0}}$(HH)$|$ is decreasing on M and is increasing on n. At $\{6 \leq M \leq 501,\ n \leq M/2\}$, the $|\overline{F_{a0}}$(HH)$|$ values belong to the range of $0.0041 \div 0.27$ a-entities at $n = 2$, of $0.011 \div 0.42$ aE at $n = 3$, of $0.017 \div 0.40$ aE at $n = 4$, of $0.078 \div 0.88$ aE at $n = 10$, and of $0.71 \div 1.10$ aE at $n = 50$.

As in case of $\overline{F_{a1}}$(d'H), the $|\overline{F_{a1}}$(HH)$|$ value is increasing on n (Figure 3b). But unlike $\overline{F_{a1}}$(d'H), the $|\overline{F_{a1}}$(HH)$|$ value is pronounced decreasing on M. For $\{6 \leq M \leq 501, 2 \leq n \leq 50,\, n < M\}$, the $|\overline{F_{a1}}$(HH)$|$ values belong to the range of $0.0041 \div 0.27$ a-entities at $n = 2$, of $0.0011 \div 0.41$ aE at $n = 3$, of $0.020 \div 0.60$ aE at $n = 4$, of $0.13 \div 4.17$ aE at $n = 10$, and of $2.65 \div 24.18$ aE at $n = 50$, being considerably smaller compared to those of $\overline{F_{a1}}$(d'H). At high values of M, the $|\overline{F_{a1}}$(HH)$|$ values are less significant. For example, for $M = 501$, the $|\overline{F_{a1}}$(HH)$|$ value is equal to 0.0041 aE at $n = 2$ and to 2.65 aE at $n = 50$.

Graphs of criteria $\overline{F_{a0}}, \overline{F_{a1}}$ and $\overline{F_{r1}}$ dependences on M and n, when using the **Adapted Sainte-Laguë method**, are largely similar to those for the Huntington-Hill method. Some quantitative differences can be found in Figures 5a-6b and Tables 2-4.

Graphs of criteria $\overline{F_{a0}}$ and $\overline{F_{a1}}$ dependences on M and n, when using the **QDLD method**, are shown în Figure 4. In these graphs, it is taken into account that at $n = 2$ and $n = 3$ the QDLD method coincides with the Sainte-Laguë one; that is, it is neutral regarding favoring the beneficiaries. That is why $4 \leq n \leq 50$.

From Figure 4a, it can be seen that out of the 58 values of the pair of sizes M and n, the largest average absolute discrepancy $\overline{F_{a0}}$(QDLD) between the largest ($i = 1$) and the smallest ($i = n$) beneficiary is obtained in case of $\{M = 201,\ n = 50\}$, this being equal to approx. -1.15 a-entities. Discrepancy $|\overline{F_{a0}}$(QDLD)$|$ almost does not depend on M and is decreasing with respect to n except for cases when $n = M-1$, in which it also little depends on n. At $\{n \leq M/2,\ 6 \leq M \leq 501\}$, this discrepancy is in the range of $0.16 \div 0.21$ a-entities at $n = 4$, of

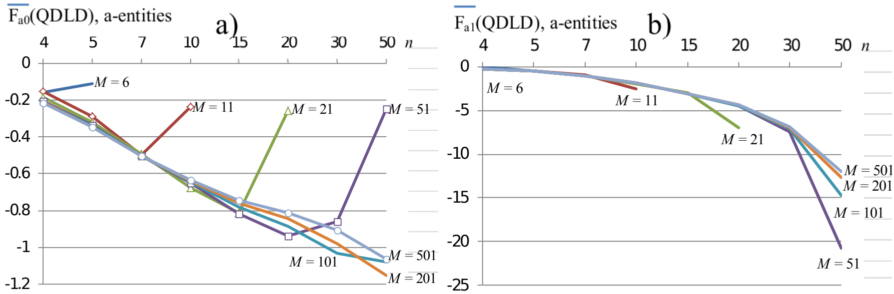

Figure 4. Criteria $\overline{F_{a0}}$ and $\overline{F_{a1}}$ dependences to $\{M, n\}$ for QDLD method.

$0.11 \div 0.35$ aE at $n = 5$, of $0.63 \div 0.67$ aE at $n = 10$, and of $0.55 \div 0.79$ aE at $n = 50$.

Figure 4b shows that the $|\overline{F_{a1}}(\text{QDLD})|$ value is ascending to n, but is slightly decreasing to M; at the same time, at $n \leq M/2$ the $|\overline{F_{a1}}(\text{QDLD})|$ value little depends on M. For $\{11 \leq M \leq 501,\ 4 \leq n \leq 50,\ n \leq M/2\}$, the $|\overline{F_{a1}}(\text{QDLD})|$ value belongs to the range of $0.24 \div 0.29$ a-entities at $n = 4$, of $0.45 \div 0.52$ aE at $n = 5$, of $1.80 \div 1.96$ aE at $n = 10$, and of $12.04 \div 14.69$ aE at $n = 50$, being considerable, especially at high values of n.

7 Comparative analyses of favoring the beneficiaries by APP methods

Let's first examine the Huntington-Hill and Adapted Sainte-Laguë methods, which guarantee the allocation of at least one a-entity to each beneficiary. For quantitative comparative estimates, Figures 5a and 5b show the graphs of the difference $\overline{F_{a1}}$(HH)-$\overline{F_{a1}}$(ASL) and those of the ratio $\overline{F_{a1}}(\text{HH})/\overline{F_{a1}}(\text{ASL})$ dependences on M and n.

From Figure 5a, it can be seen that the favoring of small beneficiaries by Huntington-Hill method is stronger than that obtained by the Adapted Sainte-Laguë one; only at values of n close to those of M it can be $\overline{F_{a1}}(\text{HH}) = \overline{F_{a1}}(\text{ASL})$. The difference $|\overline{F_{a1}}$(HH)-$\overline{F_{a1}}$(ASL)$|$ is

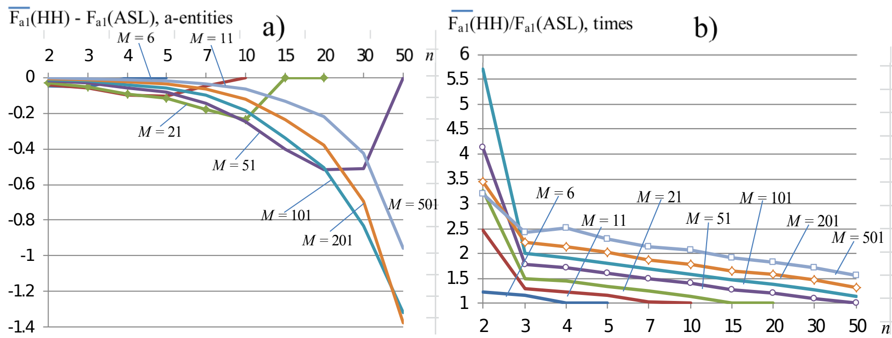

Figure 5. Difference $\overline{F_{a1}}(\text{HH})-\overline{F_{a1}}(\text{ASL})$ and ratio $\overline{F_{a1}}(\text{HH})/\overline{F_{a1}}(\text{ASL})$ dependences to $\{M, n\}$.

also small at low values of n. Thus, the difference $|\overline{F_{a1}}(\text{HH})\text{-}\overline{F_{a1}}(\text{ASL})|$ increases, and then it decreases, both with respect to n and with respect to M. The exception is only the case of $n = 2$, in which, according to calculations, this difference only decreases on M.

For $\{6 \leq M \leq 501, 2 \leq n \leq 50,\ n < M\}$, the value $|\overline{F_{a1}}(\text{HH})\text{-}\overline{F_{a1}}(\text{ASL})|$ is between $0.029 \div 0.048$ a-entities at $n = 2$, of $0 \div 1.37$ aE at $n = 50$ and of $0.96 \div 1.37$ aE at $\{n = 50,\ M = 101 \div 501\}$. The highest value of the difference $|\overline{F_{a1}}(\text{HH})\text{-}\overline{F_{a1}}(\text{ASL})|$, equal to 1.37 aE, is at $\{M = 201,\ n = 50\}$.

If, at low values of n, the difference $|\overline{F_{a1}}(\text{HH})\text{-}\overline{F_{a1}}(\text{ASL})|$ values are also small, then those of ratio $\overline{F_{a1}}(\text{HH})/\overline{F_{a1}}(\text{ASL})$, on the contrary, are relatively high (Figure 5b), reaching 5.70 times at $\{M = 101,\ n = 2\}$. The ratio $\overline{F_{a1}}(\text{HH})/\overline{F_{a1}}(\text{ASL})$ value is decreasing on n, and at $n > 2$ it is increasing on M. For example, if $M = 501$, then $\overline{F_{a1}}(\text{HH})/\overline{F_{a1}}(\text{ASL})$ value is approx.2.42 times at $n = 3$ and of approx.2.52 times at $n = 4$; also, $\overline{F_{a1}}(\text{HH})/\overline{F_{a1}}(\text{ASL}) \approx 1.57$ times at $n = 50$. For $\{6 \leq M \leq 501, 2 \leq n \leq 50,\ n < M\}$, the $\overline{F_{a1}}(\text{HH})/\overline{F_{a1}}(\text{ASL})$ value is between $1 \div 1.57$ times at $n = 50$, of $1.14 \div 1.57$ times at $\{n = 50,\ M = 101 \div 501\}$, and of $1.21 \div 5.70$ times at $n = 2$. The highest value of ratio $\overline{F_{a1}}(\text{HH})/\overline{F_{a1}}(\text{ASL})$, equal to 5.70 times, is at $\{M = 101,\ n = 2\}$. Undoubtedly, **Huntington-Hill method favors the beneficiaries stronger than the Adapted Sainte-Laguë one does**.

A clear vision of the Adapted Sainte-Laguë method superiority on non-favoring over the Huntington-Hill one is also done by the comparison based on criterion $\overline{F_{r1}}$. Figures 6a and 6b show the graphs of the difference $\overline{F_{r1}}$(HH)-$\overline{F_{r1}}$(ASL) and, respectively, of the ratio $\overline{F_{r1}}$(HH)/$\overline{F_{r1}}$(ASL) dependences on M and n.

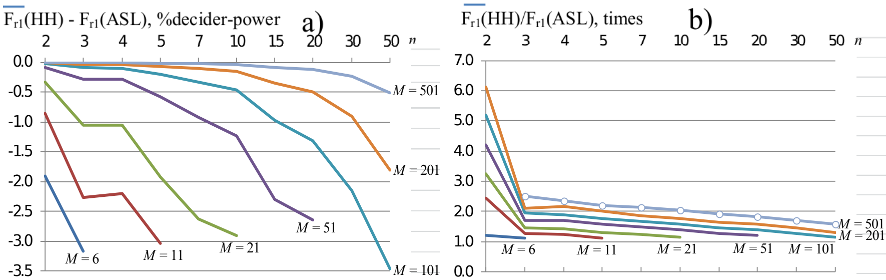

Figure 6. Difference $\overline{F_{r1}}$(HH)$-\overline{F_{r1}}$(ASL) and ratio $\overline{F_{r1}}$(HH)/$\overline{F_{r1}}$(ASL) dependences to $\{M, n\}$.

For initial data stated in Section 4, the inequality $|\overline{F_{r1}}(\text{HH})| > |\overline{F_{r1}}(\text{ASL})|$ occurs, except the cases of $n = M - 1$ for which $\overline{F_{r1}}$(HH) $= \overline{F_{r1}}$(ASL). That is, the Huntington-Hill method favors beneficiaries stronger than the adapted Sainte-Laguë one does, the difference in question reaching 3.47 %decider-power at $\{M = 101, n = 50\}$. At $\{6 \leq M \leq 501,\ n \leq M/2\}$, the difference $|\overline{F_{r1}}(\text{HH})|$-$|\overline{F_{r1}}(\text{ASL})|$ value is decreasing on M and is increasing on n (Figure 6a).

If between the case of $n = 2$ and the other cases of graphs in Figure 5a there is some discrepancy regarding the value of ratio $\overline{F_{a1}}$(HH)/$\overline{F_{a1}}$(ASL), then in case of ratio $\overline{F_{r1}}$(HH)/$\overline{F_{r1}}$(ASL) such a discrepancy is missing (Figure 6b): the increase of M always results with the increase of the ratio in question. At the same time, the ratio $\overline{F_{r1}}$(HH)/$\overline{F_{r1}}$(ASL) dependence on n is decreasing.

Although the d'Hondt method does not guarantee the allocation of at least one a-entity to each beneficiary, such as Huntington-Hill and Adapted Sainte-Laguë do, for comparison in Figure 7a the ratio $|\overline{F_{a1}}(\text{d'H})/\overline{F_{a1}}(\text{ASL})|$ dependence to M and n is given, without the cases of $M = 501$ and also those of $n = 2$.

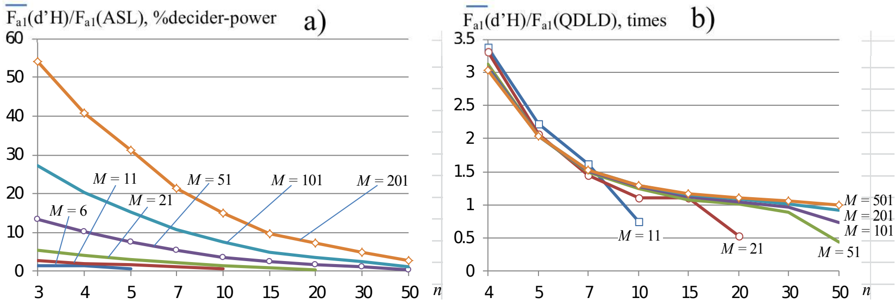

Figure 7. Ratios $\overline{F_{a1}}$(d'H)/$\overline{F_{a1}}$(ASL) and $\overline{F_{a1}}$(d'H)/$\overline{F_{a1}}$(QDLD) dependences to $\{M, n\}$.

It can be seen that the ratio $|\overline{F_{a1}}$(d'H)/$\overline{F_{a1}}$(ASL)$|$ value is decreasing on n, but is increasing on M. For $\{6 \le M \le 501, 2 \le n \le 50,\ n < M\}$, the $|\overline{F_{a1}}$(d'H)/$\overline{F_{a1}}$(ASL)$|$ values belong to the range of $0.37 \div 7.13$ times at $n = 50$, of $1.10 \div 7.13$ times at $\{n = 50,\ M = 101 \div 501\}$, of $0.63 \div 76.7$ times at $n = 5$, and of $1.42 \div 301$ times at $n = 2$. For very many cases, the value of this ratio exceeds 10 times. Values lower than 1 are obtained only for very specific cases, usually not encountered in practice: $\{M = 6, n = 5\}, \{M = 11, n = 10\}, \{M = 21, n = 15\}, \{M = 21, n = 20\}$, and $\{M = 51, n = 50\}$. It can be considered that **usually d'Hondt method favors beneficiaries much stronger than the Adapted Sainte-Laguë one does**.

It should be noted that QDLD method can be used under same conditions as the d'Hondt method. For comparison, in Figure 7b is given the ratio $|\overline{F_{a1}}$(d'H)/$\overline{F_{a1}}$(QDLD)$|$ dependence on M and n, without the cases of $n = 2$ and $n = 3$, in which the QDLD method does not favor beneficiaries, and also without the case of $M = 6$. One can observe the ratio $|\overline{F_{a1}}$(d'H)/$\overline{F_{a1}}$(QDLD)$|$ decreasing dependence on n and its weak dependence on M at relatively high values of M and n. In more detail, according to the results of calculations, compared to M, the ratio $|\overline{F_{a1}}$(d'H)/$\overline{F_{a1}}$(QDLD)$|$ value is decreasing at $n = 4$ and $n = 5$; is increasing at $n = 10$, $n = 20$, $n = 30$; and $n = 50$; and is first decreasing (at $\{n = 7, 11 \le M \le 21\}$ and $\{n = 15, 21 \le M \le 51\}$) and then is increasing.

In case of $M = 6$, the value of examined ratio is of 91.2 times at $n = 4$ and of 2.25 times at $n = 5$. For $\{11 \leq M \leq 501, 4 \leq n \leq 50,\ n < M\}$, the $|\overline{F_{a1}}(\text{d'H})/\overline{F_{a1}}(\text{QDLD})|$ values belong to the range of $0.43 \div 1.00$ times at $n = 50$, of $0.73 \div 1.00$ times at $\{n = 50, 101 \leq M \leq 501\}$, of $1.45 \div 1.62$ times at $n = 7$, and of $3.04 \div 3.38$ times at $n = 4$. Values less than 1 are obtained only for very specific cases, usually not encountered in practice: $\{M = 11, n = 10\}$, $\{M = 21, n = 20\}$, $\{M = 51, n = 30 \div 50\}$, $\{M = 101, n = 30 \div 50\}$, and $\{M = 201, n = 50\}$. Therefore, it can be considered that **usually d'Hondt method favors beneficiaries stronger than the QDLD one does**.

The character of dependences $\overline{F_{a1}}$, $\overline{F_{r0}}$, and $\overline{F_{r1}}$ of favoring the beneficiaries by the four APP methods to M and n at $n \leq M/2$ is shown in Table 2.

Table 2. Character of some dependences to M and n at $n \leq M/2$

Dependences	Apportionment methods			
	HH	ASL	d'H	QDLD
Dependence $\overline{F_{r0}}$ and $\overline{F_{r1}}$ on M	strongly decreasing	strongly decreasing	strongly decreasing	strongly decreasing
Dependence $\overline{F_{a1}}$ on M	strongly decreasing	strongly decreasing	slightly increasing	slightly decreasing
Dependence $\overline{F_{r0}}$, $\overline{F_{r1}}$ and $\overline{F_{a1}}$ on n	increasing	increasing	increasing	increasing

From Table 2, for the four examined methods, it can be seen that:

- the nature of criteria $|\overline{F_{r0}}|$ and $|\overline{F_{r1}}|$ dependences on M and n and also of the $|\overline{F_{a1}}|$ one on n is the same in all methods. Regarding the nature of the $|\overline{F_{a1}}|$ dependence on M, it coincides for the Huntington-Hill and Adapted Sainte-Laguë methods, being strongly decreasing; it slightly differs for the QDLD method, this being only slightly decreasing, and is completely different for the d'Hondt method, this being slightly increasing;

- the limits of the value ranges of $|\overline{F_{r0}}|$ and those of $|\overline{F_{a1}}|$ and $|\overline{F_{r1}}|$ for the Adapted Sainte-Laguë method are smaller than those

Table 3. Some features of APP methods regarding favoring at $n \leq M/2$

Criteria	n	Apportionment methods			
		HH	ASL	d'H	QDLD
$\lvert\overline{F_{r0}}\rvert$, %DP	2	0.0014÷10.63	0.0000 ÷ 8.72	0.18 ÷ 12.34	0
	3	0.0092÷33.36	0.0037÷30.20	0.46 ÷ 35.10	0
	4	0.027 ÷ 28.56	0.012 ÷ 25.35	0.84 ÷ 35.49	0.28 ÷ 9.02
	7	0.21 ÷ 60.21	0.12 ÷ 57.15	2.57 ÷ 52.04	1.71 ÷ 41.02
	10	0.73 ÷ 181.58	0.47 ÷ 180.11	5.14 ÷ 86.25	4.06 ÷ 108.93
$\lvert\overline{F_{a1}}\rvert$, aE	2	0.0041 ÷ 0.27	0.0013 ÷ 0.22	0.32 ÷ 0.41	0
	3	0.011 ÷ 0.42	0.0044 ÷ 0.36	0.51 ÷ 0.56	0
	4	0.020 ÷ 0.54	0.0078 ÷ 0.44	0.83 ÷ 0.87	0.0094 ÷ 0.29
	10	0.13 ÷ 1.91	0.062 ÷ 1.67	2.15 ÷ 2.34	1.80 ÷ 1.96
	50	2.65 ÷ 11.00	1.69 ÷ 9.68	10.65 ÷ 12.06	12.04 ÷ 14.69
$\lvert\overline{F_{r1}}\rvert$, %DP	2	0.0014÷10.63	0.0000 ÷ 8.72	0.18 ÷ 12.34	0
	3	0.0092÷33.36	0.0037÷30.20	0.46 ÷ 35.10	0
	4	0.010 ÷ 12.01	0.0043 ÷ 9.81	0.43 ÷ 18.50	0.14 ÷ 5.46
	10	0.065 ÷ 23.33	0.032 ÷ 20.41	1.20 ÷ 26.28	0.92 ÷ 23.91
	50	1.40 ÷ 28.80	0.89 ÷ 25.33	6.36 ÷ 27.90	6.36 ÷ 38.46

for the Huntington-Hill method. In turn, the latter are usually smaller than those for the d'Hondt method; some exceptions occur for relatively high values of n, when the requirement to allocate at least one a-entity to each beneficiary strongly influences apportionments;

- the limits of the value ranges of $\lvert\overline{F_{r0}}\rvert$ and those of $\lvert\overline{F_{a1}}\rvert$ and $\lvert\overline{F_{r1}}\rvert$ for the QDLD method in most cases are smaller than those for d'Hondt method.

The absolute degree ($\overline{F_{a1}}$) and those relative to r ($\overline{F_{r0}}$ and $\overline{F_{r1}}$) of favoring by the four APP methods are systematized in Table 3. Also, the character of dependencies $\lvert\overline{F_{a1}}(\text{HH})/\overline{F_{a1}}(\text{ASL})\rvert$, $\lvert\overline{F_{a1}}(\text{HH}) - \overline{F_{a1}}(\text{ASL})\rvert$, $\lvert\overline{F_{a1}}(\text{d'H})/\overline{F_{a1}}(\text{ASL})\rvert$, and $\lvert\overline{F_{a1}}(\text{d'H})/\overline{F_{a1}}(\text{QDLD})\rvert$ of favoring the beneficiaries by the four APP methods to M and n at $n \leq M/2$ is systematized in Table 4. The absolute comparative degree of favoring the beneficiaries by examined APP methods is systematized in Table 5. If at $n \leq M{-}1$ the value of the difference $\lvert\overline{F_{a1}}(\text{HH}){-}\overline{F_{a1}}(\text{ASL})\rvert$ is

Table 4. Character of some comparative dependencies at $n \leq M/2$

Dependencies	Comparison criteria			
	$\lvert\overline{F_{a1}}$(HH)-$\overline{F_{a1}}$(ASL)$\rvert$, a-entities	$\lvert\overline{F_{a1}}$(HH)/$\overline{F_{a1}}$(ASL)$\rvert$, times	$\lvert\overline{F_{a1}}$(d'H)/$\overline{F_{a1}}$(ASL)$\rvert$, times	$\lvert\overline{F_{a1}}$(d'H)/$\overline{F_{a1}}$(QDLD)$\rvert$, times
Dependence on M	increasing at n-10, then decreasing	increasing at $n = 2$, then decreasing	decreasing	decreasiong at $n =$ 4-5, slightly increasing at $n \geq 7$
Dependence on n	increasing	increasing	increasing	increasing

first increasing and then decreasing both on M and n (see Figure 5a), then at $n \leq M/2$ it is only increasing to n (see Table 3).

Regarding the dependence on M at $n \leq M/2$ according to the results of calculations, the difference $|\overline{F_{a1}}(\text{HH}) - \overline{F_{a1}}(\text{ASL})|$ is first increasing only in two cases, namely at $\{n = 3,\ 6 \leq M \leq 11\}$ and at $\{n = 10,\ 21 \leq M \leq 51\}$, in the other cases being only decreasing.

If the value of $\overline{F_{a1}}(\text{HH})/\overline{F_{a1}}(\text{ASL})$ ratio is significantly greater than 1, obtaining values in the range of $1.14 \div 5.70$ times, then the $|\overline{F_{a1}}(\text{d'H})/\overline{F_{a1}}(\text{ASL})|$ one can be, depending on the case, considerably higher than 1, obtaining values from 1.10 times to 301 times. That is, in terms of favoring the beneficiaries, **Huntington-Hill method yields significantly, and the d'Hondt method - considerably to the Adapted Sainte-Laguë method**. Also, **d'Hondt method yields absolutely to the QDLD one, at $n = 2$ and $n = 3$, and - significantly, in the other cases of practical interest** (only in cases of $\{M = 101,\ 30 \leq n \leq 50\}$ and of $\{M = 101,\ n = 50\}$ occurs $|\overline{F_{a1}}(\text{d'H})/\overline{F_{a1}}(\text{ASL})| < 1$). Some details on the largest and the lowest value of the difference $\overline{F_{a1}}(\text{HH}) - \overline{F_{a1}}(\text{ASL})$ are systematized in Table 6. According to Table 6, the lowest values of $|\overline{F_{a1}}(\text{HH}) - \overline{F_{a1}}(\text{ASL})|$ are obtained at $M = 501$, regardless of the value of n. At the same time, the value of M, at which the highest values of the difference $|\overline{F_{a1}}(\text{HH}) - \overline{F_{a1}}(\text{ASL})|$ are obtained, increases on n, the difference in question reaching 1.37 times at $\{n = 50, M = 201\}$ and $M/n \approx 3 \div 4$.

Table 5. The value range of comparison criteria at $n \leq M/2$

n	Comparison criteria			
	$\lvert\overline{F_{a1}}$(HH)-$\overline{F_{a1}}$(ASL)$\rvert$, a-entities	$\lvert\overline{F_{a1}}$(HH)/$\overline{F_{a1}}$(ASL)$\rvert$, times	$\lvert\overline{F_{a1}}$(d'H)/$\overline{F_{a1}}$(ASL)$\rvert$, times	$\lvert\overline{F_{a1}}$(d'H)/$\overline{F_{a1}}$(QDLD)$\rvert$, times
2	$0.0029 \div 0.048$	$1.21 \div 5.70$	$1.42 \div 301$	∞
3	$0.0063 \div 0.060$	$1.50 \div 2.42$	$1.42 \div 126$	∞
4	$0.012 \div 0.097$	$1.22 \div 2.52$	$1.88 \div 111$	$3.04 \div 3.38$
5	$0.018 \div 0.12$	$1.15 \div 2.29$	$1.51 \div 76.7$	$2.03 \div 2.23$
7	$0.033 \div 0.18$	$1.24 \div 2.14$	$2.05 \div 53.5$	$1.45 \div 1.52$
10	$0.065 \div 0.25$	$1.14 \div 2.06$	$1.29 \div 38.0$	$1.10 \div 1.30$
15	$0.13 \div 0.40$	$1.28 \div 1.91$	$2.32 \div 24.5$	$1.07 \div 1.17$
20	$0.22 \div 0.52$	$1.20 \div 1.82$	$1.74 \div 18.1$	$1.01 \div 1.11$
30	$0.43 \div 0.83$	$1.28 \div 1.71$	$2.31 \div 12.0$	$0.97 \div 1.05$
50	$0.96 \div 1.37$	$1.14 \div 1.57$	$1.10 \div 7.13$	$0.73 \div 1.001$

Table 6. The $\lvert\overline{F_{a1}}(\mathrm{HH})-\overline{F_{a1}}(\mathrm{ASL})\rvert$ largest and the lowest values at $n \leq M/2$, a-entities

	Number of beneficiaries (n)							
	2	3	4	5	7	10	20	50
M	6	11	11	21	21	51	51	201
$max\lvert\overline{F_{a1}}$(HH)-$\overline{F_{a1}}(ASL)\rvert$	0.048	0.060	0.097	0.12	0.18	0.25	0.83	1.37
M	501	501	501	501	501	501	501	501
$min\lvert\overline{F_{a1}}$(HH)-$\overline{F_{a1}}(ASL)\rvert$	0.003	0.006	0.012	0.012	0.033	0.065	0.22	0.96

8 Conclusions

Based on four notions and two issues (the fact and the quantitative estimate), there are distinguished eight aspects of favoring in apportionments. Mainly, the four quantitative aspects were explored. In this aim, five criteria were defined: the degree of favoring of large or of small beneficiaries by apportionment methods ($\overline{F_{a1}}$), the average largest absolute discrepancy of the degree of favoring between two beneficiaries ($\overline{F_{a0}}$), the average relative discrepancy between the degree of favoring of an average large beneficiary decision-maker and that of an average small beneficiary decision-maker ($\overline{F_{r1}}$), the average largest relative discrepancy of the degree of favoring between two deciders which supported different beneficiaries ($\overline{F_{r0}}$), and the largest discrepancy of the probability of favoring between two beneficiaries (F_p). The degree of favoring is measured in apportioned entities, percentage of decider-power or percentage of apportioned entities. A total of 6 APP methods are being researched, namely, Hamilton (Hare), d'Hondt (Jefferson), Huntington-Hill, Adapted Sainte-Laguë, Variable linear divisor and Quota dependent linear divisor. Hamilton method, neutral in terms of favoring, is investigated only for the purpose of comparative analysis of characteristics of other APP methods considered almost neutral on favoring.

In order to determine the values of quantitative criteria, computer simulation by SIMAP application was used for 58 variants of values for the pair $\{M, n\}$, uniform distribution of values $V_i, i = \overline{1, n}$ and sample size of 10^6. Done calculations not only confirmed some known preferences with refer to non-favoring the beneficiaries, but also permitted to estimate quantitatively the degree of favoring the beneficiaries by the 6 APP methods. For example, it was identified that:

- preferences among 6 APP methods, with refer to non-favoring of beneficiaries by criteria $\overline{F_{a1}}$, $\overline{F_{a0}}$, $\overline{F_{r1}}$ and F_p, coincide;
- the degree of favoring of beneficiaries depends both, on APP method used and on the value of initial data, and can be considerable. For example, the $\overline{F_{a1}}$(d'H) value is increasing to n and slightly increasing to M, and $9.0 \le \overline{F_{a1}}$(d'H) ≤ 12.1 (a-entities) at $\{n = 50,\ 51 \le M \le 501\}$;

- Huntington-Hill method favors small beneficiaries stronger than the Adapted Sainte-Laguë one does. The highest value of the difference $|\overline{F_{a1}}(\mathrm{HH})-\overline{F_{a1}}(\mathrm{ASL})|$, equal to 1.37 a-entities, is at $M = 201$ and $n = 50$. The highest value of ratio $\overline{F_{a1}}(\mathrm{HH})/\overline{F_{a1}}(\mathrm{ASL})$, equal to 5.70 times, is at $\{M = 101,\ n = 2\}$;
- usually d'Hondt method favors beneficiaries much stronger than the Adapted Sainte-Laguë one does. For very many cases the value of ratio $|\overline{F_{a1}}(\mathrm{d'H})/\overline{F_{a1}}(\mathrm{ASL})|$ exceeds 10 times;
- usually d'Hondt method favors large beneficiaries stronger than the QDLD method favors the small ones. In case of $M = 6$, the value of ratio $|\overline{F_{a1}}(\mathrm{d'H})/\overline{F_{a1}}(\mathrm{QDLD})|$ is of 91.2 times at $n = 4$ and of 2.25 times at $n = 5$.

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