Romanian Reports in Physics **75**, 908 (2023)

BISTABILITY EFFECTS IN MECHANICS AND ITS EXPERIMENTAL DEMONSTRATION

VITALIE CHISTOL, DUMITRU CIOCHINA, VASILE TRONCIU

Department of Physics, Technical University of Moldova, bd. Stefan cel Mare 16, Chisinau, MD-2004, Republic of Moldova
E-mail: *vasile.tronciu@adm.utm.md*

Received October 10, 2022

Abstract. We report examples of obtaining the bistability effect in mechanics and the experimental demonstration of this effect. We propose to use the graphical method of solving physics problems. The method explains qualitatively the theoretical results. We investigate the influence of different parameters on bistability properties. Finally, the experiment shows a clear evidence of bistability in a mechanical system.

Key words: bistability, hysteresis, graphical method, elastic force, Coulomb force.
DOI: https://doi.org/10.59277/RomRepPhys.2023.75.908

1. INTRODUCTION

It is well known that in case when the output parameter of a device is multivalued function of the input parameter, this device can operate in bistability mode. In this case the system can be in two stable equilibrium states, separated from each other by an unstable equilibrium state. Often bistability is followed by a hysteresis loop. This means that the transition of the system from one state to another takes place on one way, and the transition to the initial state takes place on another way. The bistability effect is well-known in optics (optical bistability) [1, 2], magnetism (magnetic bistability) [3], electricity (bistable circuits) [4], etc. In most cases the nature of this effect is quantum and its explanation goes far beyond the limits of the high school curriculum. In [5, 6] some examples of bistability in mechanics were presented. Obtaining the hysteresis loop is explained by relatively simple reasoning, not exceeding the limits of the school curriculum or the first year of university. In this paper we present some more examples in which the effect of bistability manifests itself in the processes of mechanics and we show the experimental proof of obtaining effect by one example.

2. THEORETICAL RESULTS

2.1. ROTATING THIN RUBBER RING

Let us consider a thin rubber ring with mass m and radius R_0 (see Fig. 1). The description of figures are in the text. The coefficient of elasticity of rubber is k. The ring starts to rotate in the horizontal plane around its axis with the angular velocity ω. It is necessary to find the new radius R of the ring.

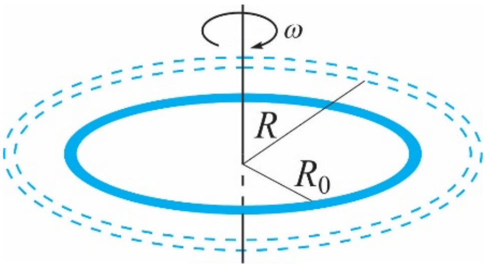

Fig. 1 – The scheme of rotating of thin ring.

A resultant centrifugal force F_r and two elastic forces F_{el} act on one half of the ring from the other half (see Fig. 2). According to Newton's first law

$$2F_{el} = F_r . \quad (1)$$

To obtain the force F_r, we divide the ring into very small elements of length dl and mass dm. A centrifugal force dF acts on each element of the ring. We can say that the ring is in a radial force field.

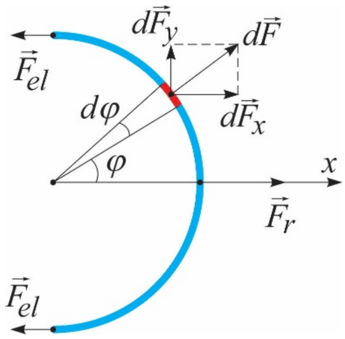

Fig. 2 – Decomposition of forces.

The force dF can be decomposed into two components $d\vec{F}_x$ and $d\vec{F}_y$. The resultant force than act on the whole ring is written as follows

$$\vec{F}_r = \int \mathrm{d}\vec{F}_x + \int \mathrm{d}\vec{F}_y. \tag{2}$$

The sum of the components $\mathrm{d}\vec{F}_y$ acting on all elements of a half-ring is equal to zero. This means that $\int \mathrm{d}\vec{F}_y = 0$ and using (2) we obtain

$$F_r = \int \mathrm{d}F_x = \int \mathrm{d}F \cos\varphi = \int \mathrm{d}m\omega^2 R\cos\varphi = \int \tau \mathrm{d}l\omega^2 R\cos\varphi\,, \tag{3}$$

where $\tau = m/2\pi R$ is the linear density of the ring material, and $\mathrm{d}l = R\mathrm{d}\varphi$. In what follows, we introduce the last expressions in (3):

$$F_r = \int \frac{m}{2\pi R} R\mathrm{d}\varphi\omega^2 R\cos\varphi = \frac{m\omega^2 R}{2\pi}\int_{-\pi/2}^{\pi/2}\cos\varphi\mathrm{d}\varphi = \frac{m\omega^2 R}{2\pi}2 = \frac{m\omega^2 R}{\pi}. \tag{4}$$

On the other hand, the elastic force has the value

$$F_{el} = kx = k\left(2\pi R - 2\pi R_0\right) = 2\pi k\left(R - R_0\right). \tag{5}$$

Finally, inserting expressions (4) and (5) into (1), we obtain the required formula for R

$$R = R_0 \Big/ \left(1 - \frac{m\omega^2}{4\pi^2 k}\right). \tag{6}$$

Let's analyze the above formula in detail. From expression (6) we observe that the obtained result is valid only for $\omega \le \omega_0 = \sqrt{4\pi^2 k/m}$. For $\omega = \omega_0$ the ring will expand to infinity (we consider rubber absolutely elastic). For $\omega > \omega_0$ solution (6) does not make any sense. On the other hand, for a better understanding of these results, we represent graphically the dependences of the forces from expressions (4) and (5) on the axis r (see Fig. 3).

We observe that, in case of $\omega < \omega_0$ for any $r < R$, F_r is greater than F_{el} and the ring will expand. For the case of $r = R$ these two forces are equal and the ring will be in a stable equilibrium, since for any $r > R$, $F_{el} > F_r$. When ω increases, the radius of ring R becomes higher. Thus, One can see in Fig. 3 that for any $\omega \ge \omega_0$ the dependencies $F_{el}(r)$ and $F_r(r)$ don't have any points of intersection. This means that for $\omega \ge \omega_0$ the ring will be expanded to infinity.

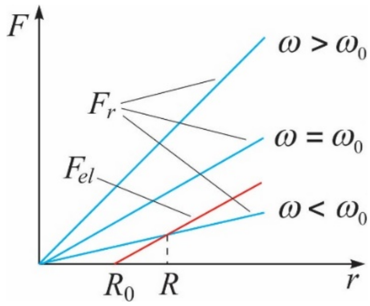

Fig. 3 – Schematic graphical solutions of equation (6).

Next, applying the proposed graphical solution method to obtain other interesting results. To do this, we slightly change the conditions of the previous problem. We consider a rigid disc of radius R_0 and a rigid ring of radius R_1, which limits the expansion of the rubber ring. Since the radius of the free rubber ring is smaller than that of the disc, the rubber ring shrinks in diameter and fits on the disc (see Fig. 4). Considering that elastic coefficient *k* is the same, and in the equilibrium position the ring is extended, the dependence of force *F* on the radius *r* in this case will have the form represented in Fig. 5.

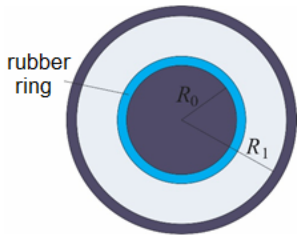

Fig. 4 – Scheme of a rigid disc and a rigid ring that limits the expansion of the rubber ring.

One can see that, in Fig. 5 that, for $\omega \le \omega_3$ the lines $F_{el}(r)$, and $F_r(r)$ have no points of intersection and $F_{el} > F_r$ for any *r*.

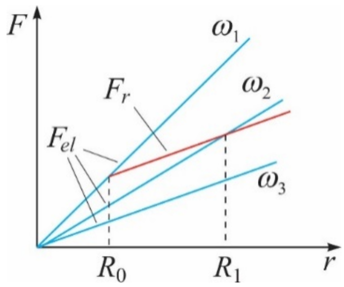

Fig. 5 – Graphical solution method applied to scheme shown in Fig. 4.

Thus, for such values of ω ($\omega \le \omega_3$) the ring will not expand. However, for $\omega_1 \le \omega \le \omega_3$ the ring will have a point of unstable equilibrium. Let us explain the reason of instability. In the case when $\omega = \omega_2$, $F_{el} > F_r$ for $r < R_1$ and $F_{el} < F_r$ for $r > R_1$. But for $\omega = \omega_2$ the ring cannot extend by itself to the radius R_1, *i.e.* only for $\omega \ge \omega_1$ the ring will expand indefinitely.

To conclude, when we start to rotate the system, the ring keeps invariant its radius until $\omega < \omega_1$. For $\omega > \omega_1$ the ring it should expand indefinitely, but its expansion is limited by the rigid ring. Thus, the expansion will stop for $r = R_1$. From the state with radius R_1 the ring can return to the state with radius R_0 only for $\omega < \omega_2$. For $\omega = \omega_1$ the ring will jump from the state with radius R_0 to the state with radius R_1, and it returns to the initial state in the case when $\omega = \omega_2$ ($\omega_2 < \omega_1$). Figure 6 shows the dependence of the radius of the ring *R* on the angular velocity of the disk. As a result, we obtained a hysteresis loop specific to the phenomenon of bistability.

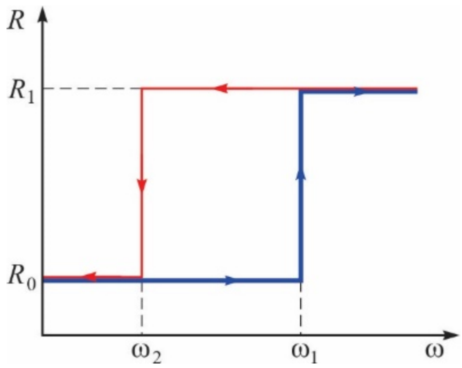

Fig. 6 – Bistability for system shown in Fig. 4.

2.2. CHARGED BALLS FIXED TO THE SPRING

In what follows, we consider two identical small balls fixed to the ends of a spring of length *l*, and elasticity coefficient *k* (see Fig. 7). The balls are charged with charge *q* equal in value and of opposite in sign. The question is the following, how much will the spring compress when they set free?

Fig. 7 – Setup of small balls fixed to the ends of a spring.

The elastic force of the spring acts on a ball

$$F_1 = kx\,, \tag{7}$$

and the Coulomb force has the following absolute value

$$F_2 = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{\left(l-x\right)^2}\,. \tag{8}$$

The balls are in equilibrium if these two forces are equal. We equate the expressions on the right sides of formulas (7) and (8) and obtain

$$k\Delta x = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{\left(l-x\right)^2}\,,$$

or final formula

$$\Delta x\left(l-x\right)^2 = \frac{q^2}{4\pi\varepsilon_0 k}\,. \tag{9}$$

We obtained an cubic equation, which has three solutions. The question is the following, can the balls to have three equilibrium positions *i.e.* three real solutions? Equation (9) cannot be solved analytically (at least, within the limits of the school curriculum or first university year program), but it can be solved graphically. Additionally, from the graphs we can distinguish how many solutions are obtained from (9).

Figure 8 shows the graphs obtained with help of expressions (7) and (8). Blue line 1 represents the dependence of the elasticity force (7), and red curves 2 of the Coulomb force (8) on extension x, respectively.

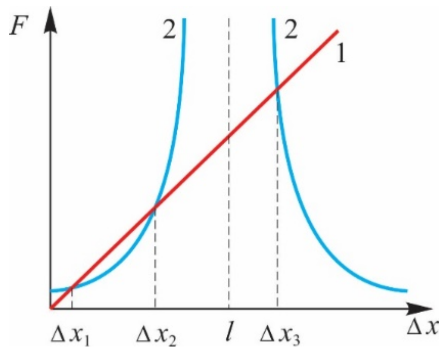

Fig. 8 – Schematic solutions of equations (7) and (8).

One can see that, the solution x_3 cannot has any physical sense in the condition of our problem, since in this position the compression of the spring is always greater than its length *l*. However, the solution x_3 can has a physical meaning only in the case when the charges of the balls have the same sign. The other two solutions x_1 and x_2 have the physical meaning. Initially, for $x = 0$, the Coulomb force is greater than the elastic force and the spring will compress to the equilibrium position $x = x_1$. For a compression greater than x_1 the elastic force becomes greater than the Coulomb force and the spring returns to the equilibrium state. Thus, in the position $x = x_1$ the spring is in a state of stable equilibrium. Compressing the spring further, we reach the other equilibrium position $x = x_2$. For $x > x_2$ the spring suddenly compresses to a position determined by the dimensions of the spring. So, in position x_2 the spring is in an unstable state of equilibrium. Solving the problem graphically, we can determine the limit values of the elasticity coefficient or the ball charges for which the problem still has solutions.

In what follows, let us investigate the influence of varying the charges of balls on features of system shown in Fig. 7. When we increase the charges of balls, the curves 2 from Fig. 9 moves up and the equilibrium positions x_1 and x_2 approach each other. Let us look into details. As shown in Fig. 9 for a certain value of the ball charge $q = q_0$ the equilibrium positions merge into one there is only one solution $x = x_0$. By further increasing the charges of balls, the Coulomb force will always be greater than the elastic force and the spring will not have any equilibrium state. For $q > q_0$ the spring will compress to the position $x = x_4$ determined by the dimensions of the spring. By decreasing the charges on the balls, they remain in the same position until a value the charges $q = q_2$. For $q = q_2$ the balance of the charge becomes unstable and it jumps to the stable equilibrium position $x = x_5$.

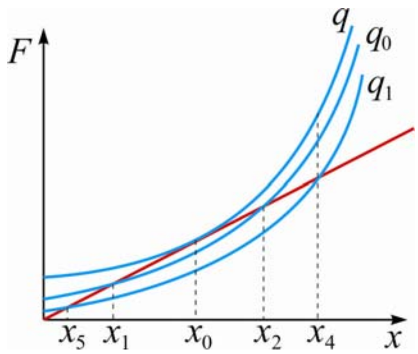

Fig. 9 – The influence of variation of the charges of balls on features of system shown in Fig. 7.

The dependence of spring compression on ball charge is shown in Fig. 10. One can observe that, in this case we also obtain a hysteresis loop specific to the phenomenon of bistability.

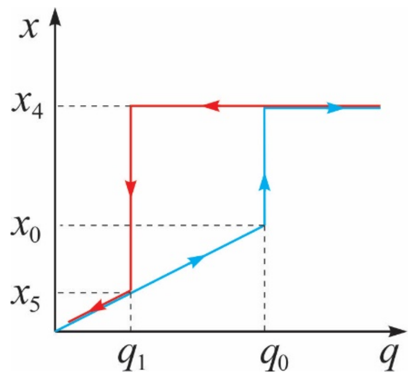

Fig. 10 – Bistability of setup shown in Fig. 7.

2.3. ROTATING DISK WITH A BODY FIXED TO A SPRING

A similar to bistability effect is obtained also in the following example. We consider a disk that can rotate with an angular velocity ω (see Fig. 11). One end of a spring is fixed to the disc and other one to a small body of mass m. The spring has a length l_0 and the body is placed at distance x_0 from axis of the disc. The spring expands and body is fixed at distance x_1 from the axis of disc, so that its movement is limited by limiters 1 and 2. The coefficient of elasticity of the spring is k. Let us calculate angular velocity ω of the disk when the body has to be at distance $x_1 \geq x \leq x_2$.

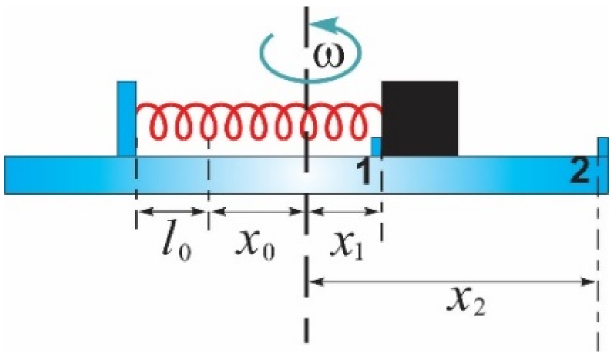

Fig. 11 – Schematic view of disk that can rotate with an angular velocity.

Two forces act on the body (apart from the reaction forces of the limiters)

$$F_1 = k\left(x + x_0\right), \tag{10}$$

$$F_2 = m\omega^2 x\,. \tag{11}$$

By equalizing these two forces, we obtain the relation for the distance x

$$x = \frac{kx_0}{m\omega^2 - k} . \tag{12}$$

From (12) we observe that the obtained result is valid only for $\omega \geq \omega_0 = \sqrt{k/m}$. Figure 12 shows the dependences F_1 (x) and F_2 (x).

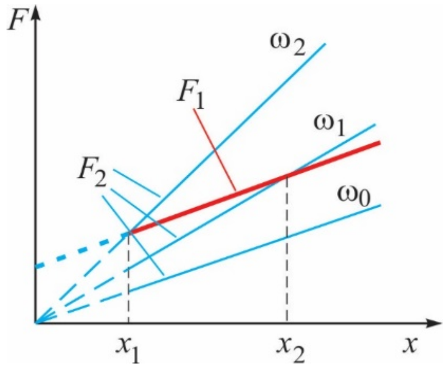

Fig. 12 – Dependence of forces F_1 (elastic) and F_2 (centrifugal) on distance x.
F_2 is calculated for different angular velocity ω.

Figure 12 shows that for any angular velocity smaller than ω_2 the force $F_2 > F_1$, and the reaction force of the limiter 1 keeps the body at rest at the distance r_1. On the other hand for $\omega = \omega_2$ the system moves to metastable state, *i.e.* the spring begins to expand and the body moves away to distance $x = x_2$. When increasing the angular velocity ω, limiter 2 keeps the body at this distance. By decreasing the angular velocity ω of the disc, the body will be kept at this distance up to the velocity $\omega_1 < \omega_2$. For $\omega = \omega_1$ the body moves (by jumping) in the initial position x_1. A clear evidence of bistability is obtained similar to that of Fig. 6.

3. EXPERIMENTAL CONFIRMATION

Finally we experimentally verified the achievement of the bistability effect in the case similar to that presented in the example from Section 2.3. In our experiment, on a disk was fixed a small body in elastic wire attached by a red LED (see Fig. 13). The body can move along the radius of the disc and its movement is limited by limiters 1 and 2. In the initial position the elastic is extended and the body is in position 1. The experiment was filmed and placed on YouTube [7].

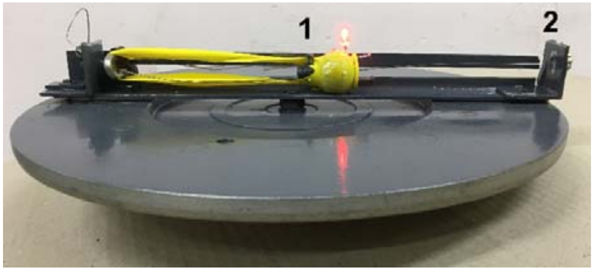

Fig. 13 – Experimental setup.

From video one can see that, by increasing the angular velocity of the disc, the body remains in the position 1. At the angular velocity ω_1 approximately equal to 9.8 Hz, the body moves to position 2. By decreasing the angular velocity of the disc, the body remains in the same position. At a angular velocity ω_2 approximately equal to 7.2 Hz the body returns to its initial position. Thus, the body describes the hysteresis loop shown in Fig. 6.

4. CONCLUSIONS

In conclusion, we mention that the graphical method presented in this paper is a very important tool for understanding the effect of bistability in mechanics. The results presented can be used to significantly improve the understanding of this phenomenon by students in schools and first year of University, and for explaining the physical processes that take place. We denote that in the case of non-linear dependencies between physical quantities, we have to take into account that the effect of bistability can occur. Thus, to avoid mistakes in solving physics problems it is necessary to consider this phenomenon. We believe that our work provides a good basis for future, more detailed studies of bistability in systems by graphical method. The presented results may be used to significantly improve the understanding of phenomenon of bistability.

Acknowledgements. Authors acknowledge the support from the project 20.800 09.5007.08.

REFERENCES

1. H.M. Gibbs, *Optical Bistability: Controlling Light With Light*, Academic Press, 1985.
2. L. Lugiato, F. Prati, M. Brambilla Nonlinear, *Optical Systems*, Cambridge University Press, 2015.
3. E. Della Tore, *Magnetic Hysteresis*, Wiley-IEEE Press, 1998.
4. K. Brindley, *Starting Electronics*, Elsevier, Third edition, 2005.
5. V. Chistol, C. Pârtac, and N. Ungureanu, *Applying physics problems to students technological education* (in Romanian), The Technical-Scientific Conf. Students and Doctoral Students Dedicated to the Year of Physics, 17 Noiembrie 2005, Chişinău, UTM, pp. 47–48.
6. Vitalie Chistol, Vasile Tronciu, *Graphical method of solving problems on bistability in physical systems*, European Journal of Physics **42**(6), 065002 (2021), DOI: 10.1088/1361-6404/ac1447.
7. https://www.youtube.com/watch?v=JJpJE278hgY&ab_channel=PotatoTroll