Preradicals and closure operators in modules: comparative analysis and relations

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The theory of radicals in modules is based by the notion of *preradical* (as subfunctor of identical functor) [1]. The other important notion of the modern algebra is the *closure operator* (as a function C which by every pair of modules $N \subseteq M$ defines a submodule $C_M(N) \subseteq M$, C being compatible by the morphisms of R-Mod) [2]. The purpose of this communication consists in the elucidation of the relations between these fundamental notions and the comparison of results of those respective theories. The closure operators in some sense are the generalization of preradicals, since the class $\mathbb{PR}(R)$ can be inserted in $\mathbb{CO}(R)$ (by two methods). This important fact determines a close connection between the results of the respective domains.

In particular, there exists some correspondences between the main types of preradicals of R-Mod and respective types of closure operators. Also there exists a remarkable connection between the operations in the big lattices $\mathbb{PR}(R)$ and $\mathbb{CO}(R)$. These facts show the parallelism and similarity of two theories. However, it is obvious that $\mathbb{CO}(R)$ is essentially „larger" than $\mathbb{PR}(R)$ (closure operators are the functions of two variables). Therefore in the study of closure operators we must apply both the classical methods of radical theory, adding the constructions, notions and applications, related by the specificity of closure operators.

In continuation we formulate some typical results of this domain.

Theorem 1. *There exists a monotone bijection between:*
a) the maximal closure operators of R-Mod and the preradicals of R-Mod: $Max[\mathbb{CO}(R)] \cong \mathbb{PR}(R)$;
b) the minimal closure operators of R-Mod and the preradicals of R-Mod: $Min[\mathbb{CO}(R)] \cong \mathbb{PR}(R)$;
b) the equivalence classes of $\mathbb{CO}(R)$*, determined by the relation „$\sim$"* $(C \sim D \overset{def}{\Longleftrightarrow} r_C = r_D)$*, and the preradicals of R-Mod:* $\mathbb{CO}(R)/_\sim \cong \mathbb{PR}(R)$.

Theorem 2. *There exists a monotone bijection between:*
a) the idempotent preradicals of R-Mod and the closure operators

which are maximal and weakly hereditary;
b) the radicals of R-Mod and the closure operators which are maximal and idempotent;
c) the pretorsions (torsions) of R-Mod and the closure operators which are minimal and hereditary (maximal, idempotent and hereditary).

Theorem 3. *There exists a monotone bijection between the cohereditary closure operators of R-Mod and the ideals of R.*

Bibliography

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