Relatively hereditary radical classes of rings

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A (Kurosh-Amitsur) *radical class* of rings is a non-empty homomorphically closed class $\mathcal{R}$ such that for each ring A,
$\sum\{I \triangleleft A : I \in \mathcal{R}\} \in \mathcal{R}$ and
$\mathcal{R}(A/\mathcal{R}(A)) = 0$.

Examples *Jacobson* radical class consists of all quasiregular rings $((\forall a)(\exists b)a + b + ab = 0)$.
Nil radical class is the class of nil rings $((\forall a)(\exists n)a^n = 0)$.
Idempotent radical class E consists of all rings A for which $A^2 = A$.

The first two are *hereditary*: they contain all ideals of all their members. Considerable work has been done on radical classes $\mathcal{R}$ with even stronger closure properties, e.g. being *left-hereditary* ($A \in \mathcal{R} \Rightarrow$ $\mathcal{R}$ contains all left ideals of A) and *strongly hereditary* ($A \in \mathcal{R}$ implies $\mathcal{R}$ contains all subrings of A).

There has been little attention to radical classes which are hereditary for some ideals but not (necessarily) all ideals. For instance nothing seems to be known about radical classes which are hereditary for *maximal* ideals, or for *prime* ideals.

A radical class $\mathcal{R}$ is hereditary if and only if $\mathcal{R}(I) = I \cap \mathcal{R}(A)$ for every ideal I of every ring A; [1], [2], p.46. We have some generalizations of this. For a non-empty class $\mathcal{C}$ of non-zero rings which is hereditary for non-zero ideals, we call an ideal I of a ring A a $\mathcal{C}$-*ideal* if $A/I \in \mathcal{C}$. For example when $\mathcal{C}$ is the class of simple (resp.

prime, resp. non-zero finite) rings, the $\mathcal{C}$-ideals are the maximal (resp. prime, resp. finite index) ideals.

For any class $\mathcal{C}$ as described, a radical class $\mathcal{R}$ is hereditary for $\mathcal{C}$-ideals if and only if $\mathcal{R}(I) = I \cap \mathcal{R}(A)$ for every $\mathcal{C}$-ideal I of every ring A.

This result is not definitive: the *essential* ideals satisfy its conclusion but these are not the $\mathcal{C}$-ideals for any $\mathcal{C}$.

If $\mathcal{C}$ is also closed under non-zero homomorphic images, then for every class $\mathcal{M}$ which is hereditary for $\mathcal{C}$-ideals, so is its lower radical class. (This generalizes a well known property of hereditary classes. [3], [2], p.49.)

The class $\mathcal{D}^*$ of rings with divisible additive groups is a radical class which is not hereditary (for all ideals) but is hereditary for maximal and for prime idels and vacuously for ideals of finite index. Hence for every hereditary radical class $\mathcal{R}$ $\mathcal{R} \cap \mathcal{D}^*$ has these relatively hereditary properties and in many cases is not hereditary, though sometimes it is, e.g. when $\mathcal{R}$ is the class of *regular* rings.

Bibliography

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