Idempotents and Radical Classes of Rings

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A (Kurosh–Amitsur) *radical class* of rings is a non-empty homomorphically closed class $\mathcal{R}$ such that

(i) $\forall\, A$, $\sum\{I \triangleleft A:\ I \in \mathcal{R}\} \in \mathcal{R}$;

(ii) $\forall\, A$, $\mathcal{R}(A/\mathcal{R}(A)) = 0$.

Examples

Jacobson radical class: the class $\mathcal{J}$ of *quasiregular* rings $[(\forall\, a)(\exists\, b)(a + b + ab = 0)]$.

Nil radical class: class of *nil* rings $[(\forall\, a)(\exists\, n)(a^n = 0)]$.

Idempotent radical class: class of rings A for which $A^2 = A$.

Clearly if $I \triangleleft A$ and $I \in \mathcal{R}$, then $I \subseteq \mathcal{R}(A)$. The analogous statements for left ideals and subrings are not true in general, but much work has been done on radical classes for which they *are* true.

Our concern is with *corners*, subrings of the form eAe, where e is an idempotent in a ring A, and it is instructive to compare radical theoretic results involving corners with analogous results involving other types of subrings. We call a radical class $\mathcal{R}$

- *corner-hereditary* if it satisfies $e^2 = e \in A \in \mathcal{R} \Longrightarrow eAe \in \mathcal{R}$;
- *very corner-hereditary* if it satisfies $e^2 = e \in A \Longrightarrow \mathcal{R}(eAe) = eAe \cap \mathcal{R}(A)$;
- *corner-strict* if it satisfies $e^2 = e \in A \& eAe \in \mathcal{R} \Longrightarrow eAe \subseteq \mathcal{R}(A)$.

Very corner-hereditary radical classes are corner-hereditary but not conversely in general. The radical class of strongly regular rings is corner-hereditary but not very corner-hereditary. $\mathcal{J}$ is very corner hereditary; $\mathcal{N}$ is corner-hereditary, but whether it is very corner-hereditary or not is equivalent to the *Köthe Problem*. If $\mathcal{R}$ satisfies the condition

$$e^2 = e \in A \& \mathcal{R}(A) = 0 \Longrightarrow \mathcal{R}(eAe) = 0$$

then $\mathcal{R}$ is corner-strict but the converse is false. This parallels known results for left strong radical classes and left ideals (cf. also strict radical classes and subrings).

Every radical class is hereditary for the corners corresponding to *central* idempotents as these are direct summands and hence homomorphic images. Less obviously, all radical classes are hereditary for the corners corresponding to *left semicentral* idempotents (those for which $Ae - eAe$). Nevertheless these idempotents are relevant to some other constructions which yield radical classes. For certain properties (P) of idempotents, the class of rings A for which every non-zero homomorphic image has a non-zero idempotent with (P) is a radical class. When (P) is the property of simply being an idempotent, we get the Brown–McCoy radical, while for central idempotents we get the lower radical class defined by the class of rings with identity.