ON THE PRE-ORDERING OF AUTOMORPHIC LOOPS AND MOUFANG LOOPS

VASILE I. URSU

Communicated by Lucian Beznea

It is proved that if an automorphic nilpotent loop (nilpotent Moufang loop) does not have finite order elements, then it is preordered.

AMS 2010 Subject Classification: 20N05.

Key words: loop, partially ordered, associator, commutator.

1. INTRODUCTION

A set L endowed with three operations: multiplication, right division, and left division, denoted by $\cdot$, $/$. $\backslash$, is called a loop if $(L, \cdot)$ is a groupoid with unit, $1 \in L$, and $x/y \cdot y = xy/y = y \cdot (y\backslash x) = y\backslash(yx) = x$ for all x, y of L. A loop L is called partially ordered if L is endowed with a partial order $(L, \leq)$ such that if $x \leq y$ then $xz \leq yz$, $x/z \leq y/z$, $zx \leq zy$ and $z\backslash x \leq z\backslash y$. If $\leq$ is actually a total order then the loop L is said to be linearly ordered. A loop is said to be *pre-ordered* if any partial order can be extended to a linear order. In connection with the study of partially ordered nilpotent loops, the question naturally arises when these loupes are pre-ordered. This question was solved for nilpotent groups. E.P. Shimbareva [1] showed that a partially ordered abelian group can be pre-ordered up if it does not contain elements of finite order. However, in [2], A.I. Maltsev proved the theorem on the pre-ordered holds for nilpotent groups as well as for locally nilpotent groups without elements of finite order. In another way, this result was proved by A.H. Rhemtulla [3].

We show that the theorem on the property of pre-ordering takes place for a wider class – the class of partially ordered locally nilpotent automorphic loops (respectively Moufang loops) without elements of finite order.

We recall several definitions, results and notation. Some of them can be found in [4], and also in [5].

For any elements x, y, z of loop L, the *right associator* (x, y, z), the *left associator* $[x, y, z]$ and the *commutator* $[x, y]$ are defined by the equalities

$$(x, y, z) = x\backslash((xy \cdot z)/yz),\ [x, y, z] = ((xy)\backslash(x \cdot yz))/z$$

and

$$[x,y] = x\backslash(y\backslash[x,y]) \ \ [5, VIU].$$

The *group of inner mapping group* $J(L)$ of the loop L is generated by all substitutions of the form

$$R_{x,y} = R_x R_y R_{xy}^{-1}, \ \ L_{x,y} = L_x L_y L_{yx}^{-1}, \ T_x = R_x L_x^{-1} \ \ (x, y \in L),$$

where

$$xL_y = yR_x = xy \ \ [4].$$

If all permutations of $J(L)$ are automorphisms, then L is an automorphic loop (or A-loop) [6]. The loop in which identity $x(y \cdot zy) = (xy \cdot z)y$ is true is called the Moufang loop.

The subloop H of the loop L is *normal* in L if one of the following three equivalent conditions is satisfied

(i) $H \cdot xy = Hx \cdot y, \ xy \cdot H = x \cdot yH, \ xH = Hx;$

(ii) $HR_{x,y} = H, \ HL_{y,x} = H, \ HT_x = H';$

(iii) $[H, L] \subseteq H,$

where $[H, L]$ is the subloop of L generated by all (left and right) associators and commutators of the form (a, x, y), $[x, y, a]$, $[a, x]$, for any $a \in H$ and for any $x, y \in L$ [4,5]. A normal subloop of a loop L will be called *invariant subloop* or a *normal divisor* of a loop L.

The subloop of the loop L generated by all (left and right) associators and commutators of the loop L is called the associate-commutator and is denoted by L'. The center of a loop L is a subset

$$Z(L) = \{a \in L \mid \ ax \cdot y = a \cdot xy, \ x \cdot ya = xy \cdot a, \ \ ax = xa \ \text{for any} \ \ x, y \in L\}.$$

It is not difficult to verify that the associate-commutator L' and the center $Z(L)$ are normal subloops of a loop L. A series of subloops

$$L = L_0 \supseteq L_1 \supseteq \ ... \supseteq L_n = \{1\}$$

is called a central series of the loop L if each subloop L_i is normal in L_{i-1}, and all of its factor-loops L_{i-1}/L_i are central, *i.e.*

$$L_{i-1}/L_i \subseteq Z(L/L_i) \ \ \text{for all} \ \ i \leq n$$

or, equivalently,

$$[L_{i-1}, L] \subseteq L_i \ \ \text{for all} \ \ i \leq n,$$

where $[H, L]$ is the subloop of loop L generated by all (left and right) associators and commutators of the form where (a, x, y), $[x, y, a]$, $[a, x]$, where $a \in H$, $x, y \in L$. A loop having a central series with a finite number n of a subloop is said to be (*central-*) *nilpotent*, and the smallest such number n is called the *nilpotency class.* It can be seen directly from the definition that the nilpotent group consists of a class intermediate between the class of abelian groups and the class of nilpotent loops, and the Abelian groups are nilpotent loops of class 1.

2. MAIN RESULT

We need some statements from [5, Theorem 1] and from [7, Theorem 1], which we formulate under one sentence.

PROPOSITION 1. *A finitely generated nilpotent automorphic loop (resp. Moufang loop) L satisfies the maximality condition, that is, every subloop of the loop L has a finite number of generators.*

We shall also use the following assertion, which follows easily from ([5], resp. [7]).

PROPOSITION 2. *If L is a nilpotent automorphic loop (resp. Moufang loop) of class n, then for any $h, h' \in L_{n-2}$ $x, x', y, y' \in L$ the following equalities hold:*

$$(h, x, y) = [y, x, h]^{-1};$$
$$(hh', x, y) = h, x, y)(h', x, y);$$
$$(x, y, hh') = (x, y, h)(x, y, h');$$
$$(h, xx', y) = (h, x, y)(h, x', y);$$
$$(h, x, yy') = (h, x, y)(h, x, y');$$
$$(h, x, y) = (y, x, h)^{-1};$$
$$(h, x, y) = (x, h, y)(h, y, x) \quad (resp., \ (h, x, y) = (x, y, h));$$
$$[hh', x] = [h, x][h', x] \quad (resp., \ [hh', x] = [h, x][h', x](h, h', x)^3);$$
$$[h, xx'] = [h, x][h, x'] \quad (resp., \ [h, xx'] = [h, x][h, x'](h, x, x')^3.$$

A groupoid A of the loop L will be called invariant in L if the equalities are true

$$xA = Ax, \ x \cdot yA = xy \cdot A, \ Ax \cdot y = A \cdot xy, \text{ for any } x, y \in L.$$

LEMMA 3. *Let H – be a normal subloop of a nilpotent automorphic loop (respectively, a nilpotent Moufang loop) L with a finite number of generators, and A a invariant groupoid in L contained in H and contains the unit. If a* *suitable positive power of each element of the loop H is contained in the groupoid $A[H, L]$, then some positive power of each element of H is contained in A.*

Proof. We determine inductively the series of subloops

$$H^{(0)} \supseteq H^{(1)} \supseteq ... \supseteq H^{(i)} \supseteq ...$$

where $H^{(0)} = H,\ H^{(i)} = [H^{i-1}, L]$ at $i \geq 1$. Since H is a normal subloop, then $H^{(1)} = [H, L] \subseteq H$, therefore $[H^{(1)}, L] \subseteq [H, L] = H^{(1)}$, *i.e.* $H^{(1)}$ is a normal subloop of a loop L. Further, by induction, we verify that all the subloops $H^{(i)}$ are normal in L. From the nilpotency of the loop L it follows that for some s there is $H^{(s)} = 1$. If $s = 1$, then H is a central subloop of L and the assertion of the lemma is trivial. Therefore, suppose that the lemma is valid for all automorphic nilpotent loops (respectively, nilpotent Moufang loops) and their subloop for which we consider an automorphic nilpotent loop (respectively, nilpotent Moufang loop) L and its subloop that satisfy the conditions of Lemma 3 such that $H^{(s+1)} = 1$.

We denote by $\bar{A}$ and $\bar{H}$ the images in the factor-loop $\bar{L} = L/H^{(i)}$. By the induction hypothesis, we conclude that for each element $h \in H$ there is a positive number L such that $h^l = a \cdot z$, where $a \in A$ and $z \in H^{(s)}$. By Proposition 1, the subloop $H^{(s-1)}$ of L has a finite number of generators. Let h_1, ..., h_q – the generators of the loops $H^{(s-1)}$ and g_1, ..., g_p – be the generating loops of L. According to what has been said, we have $h_i^l = a_i u_i$ where $a_i \in A,\ u_i \in H^{(s)}\ (i = 1, ..., q)$.

It follows from the condition $H^{(s+1)} = 1$ that the elements $H^{(s)}$ lie in the center $Z(L)$ of the loop L. Since $H^{(s)} = [H^{(s-1)}, L] \subseteq Z(L)$, each element $z \in H^{(s)}$ is the product of associators and commutators of the form $[h, x, y]$, $[x, y, h]$ and $[h, x]$, where $h \in H^{(s-1)}$, $x, y \in L$. Using the identities in Proposition 2, we can represent $z \in H^{(s)}$ in the form

$$z = \prod_{\substack{1 \leq i \leq q \\ 1 \leq j, k \leq p}} (h_i, g_j, g_k^{\alpha_{ijk}}) \cdot \prod_{\substack{i \leq 1 \leq q \\ 1 \leq j \leq p}} [h_i, g_j^{\beta_{ij}}]$$

(*resp.*,

$$z = \prod_{\substack{1 \leq i \leq q \\ 1 \leq j < k \leq p}} (h_i, g_j, g_k^{\alpha_{ijk}}) \cdot \prod_{\substack{i \leq 1 \leq q \\ 1 \leq j \leq p}} [h_i, g_j^{\beta_{ij}}]).$$

Since all the factors of the right-hand side of the last equality are central elements, we obtain

$$\begin{aligned}
z^l &= \prod_{\substack{1 \le i \le q \\ 1 \le j,k \le p}} (h_i, g_j, g_k^{\alpha_{ijk}})^l \cdot \prod_{\substack{i \le 1 \le q \\ 1 \le j \le p}} [h_i, g_j^{\beta_{ij}}]^l \\
&= \prod_{\substack{1 \le i \le q \\ 1 \le j,k \le p}} (h_i^l, g_j, g_k^{\alpha_{ijk}}) \cdot \prod_{\substack{i \le 1 \le q \\ 1 \le j \le p}} [h_i^l, g_j^{\beta_{ij}}] \\
&= \prod_{\substack{1 \le i \le q \\ 1 \le j,k \le p}} (a_i u_i, g_j, g_k^{\alpha_{ijk}}) \cdot \prod_{\substack{i \le 1 \le q \\ 1 \le j \le p}} [a_i u_i, g_j^{\beta_{ij}}] \\
&= \prod_{\substack{1 \le i \le q \\ 1 \le j,k \le p}} (a_i, g_j, g_k^{\alpha_{ijk}}) \cdot \prod_{\substack{i \le 1 \le q \\ 1 \le j \le p}} [a_i, g_j^{\beta_{ij}}]
\end{aligned}$$

(*resp.*,

$$\begin{aligned}
z^l &= \prod_{\substack{1 \le i \le q \\ 1 \le j < k \le p}} (h_i, g_j, g_k^{\alpha_{ijk}})^l \cdot \prod_{\substack{i \le 1 \le q \\ 1 \le j \le p}} [h_i, g_j^{\gamma_{ij}}]^l \\
&= \prod_{\substack{1 \le i \le q \\ 1 \le j < k \le p}} (h_i^l u_i, g_j, g_k^{\alpha_{ijk}}) \cdot \prod_{\substack{i \le 1 \le q \\ 1 \le j \le p}} [h_i^l u_i, g_j^{\gamma_{ij}}] \\
&= \prod_{\substack{1 \le i \le q \\ 1 \le j < k \le p}} (a_i u_i, g_j, g_k^{\alpha_{ijk}}) \cdot \prod_{\substack{i \le 1 \le q \\ 1 \le j \le p}} [a_i u_i, g_j^{\gamma_{ij}}] \\
&= \prod_{\substack{1 \le i \le q \\ 1 \le j < k \le p}} (a_i, g_j, g_k^{\alpha_{ijk}}) \cdot \prod_{\substack{i \le 1 \le q \\ 1 \le j \le p}} [a_i, g_j^{\gamma_{ij}}]).
\end{aligned}$$

Note $a_0 = a_1 a_2 ... a_q$, where $a_1, a_2, ..., a_q$ are the above elements of A. But since A is an invariant groupoid, for any $a \in A$ associators and commutators of the form $[a, x, y]$, $[x, y, a]$ and $[a, x]$ belong to A, it follows from the last equalities that $z^l \in A$. Therefore, we conclude that for every element $h \in H$ there exists a positive number l such that $h^l = a \cdot z$, where $a \in A$ and $h' \in H^{(s)}$, and for which

$$h^{l^2} = (h^l)^l = (az)^l = a^{l^2} z^l$$

and $h^{l^2} \in A$. $\square$

Theorem 4. *Every locally nilpotent automorphic loop (respectively, locally nilpotent Moufang loop) L without elements of finite order is pre-ordered.*

Proof. Suppose first that the loop L has a finite number of generators. We denote by A the collection of all elements of L greater than or equal to one.

A is an invariant groupoid in a loop L. Let H be the collection of those elements of the loop L, some positive degree and some negative degree of which are contained in $A[L, L]$.

We show that H is a normal subloop of L. Indeed, if $x, y \in H$ then there exists $n \geq 1$ such that $x^n, y^n, y^{-n} \in A[L, L]$, where y^{-1} we denote the element $1/y$. Then we have

$$(x \cdot y)^n \in x^n \cdot y^n [L, L] \subseteq x^n \cdot y^n A[L, L] \subseteq A[L, L],$$

$$(x/y)^n \in (xy^{-1})^n [L, L] \subseteq x^n \cdot y^{-n} A[L, L] \subseteq A[L, L],$$

$$(y \backslash x)^n \in (x/y)^n [L, L] \subseteq A[L, L].$$

Consequently $x \cdot y, x/y, y \backslash x \in H$, *i.e.* H is the subloop. Since $[L, L] \subseteq H$ then H is a normal subloop in the loop L.

We now show that H is a convex subloop. Let $h \in H$ and $1 < x < h$. Then $1 < x < h$, $1 < h \cdot x^{-1}$ and $y = h \cdot x^{-1} \in A$. By assumption, for some natural number n we have $h^{-n} \in A[L, L]$ and since $y = h \cdot x^{-1}$, then

$$\begin{aligned} x^{-n} &= (x^{-1})^n = (1 \cdot x^{-1})^n = (h^{-1} h \cdot x^{-1})^n \\ &= (h^{-1}(h^{-1}, h, x^{-1}) \cdot hx^{-1})^n \in (h^{-1}[L, L] \cdot y)^n \\ &= (h \cdot y)^n [L, L] = ((h^{-n} \cdot y^n)[L, L] = h^{-n} \cdot y^n [L, L] \\ &= h^{-n} A[L, L] \subseteq A[L, L] \cdot A[L, L] = A[L, L]. \end{aligned}$$

And so $x \in A$ and $x^{-1} \in A[L, L]$, *i.e.* $x \in H$.

From the definition of H it is clear that if for some $x \in L$ and $x^m \in H$, then for some natural number n we have $x^{mn} \in A[L, L]$ and $x \in H$. Hence the factor-loop L/H does not contain non-unit elements of finite order . Further, $L \neq H$, since from the equality $L = H$, by Lemma 3, it follows that for each there $a \in A$ exists a natural number n such that $a^{-n} \in A$, that is, $a^{-n} \geq 1$, which is impossible. We now $H_1 = H$ and then construct the chain of convex normal subloops in the following way. We now assume $H_1 = H$ and beyond we construct a chain of convex normal subloops in the following way. Let the normal subloop H_i be built. We denote A by the set of elements greater than or equal to one. Obviously $A_i = A \cap H_i$, so that A_i is invariant groupoid in L. We denote by H_{i+1} the collection of elements of H_i, some positive and some negative degree of which is contained in $A_i[H_i, L]$. It is easy to see H_{i+1} – that is a convex normal subloop of a loop L and that the factor-loop H_i/H_{i+1} is torsion-free. By Lemma 3, the equality $H_{i+1} = H_i$ is excluded and takes place a strict inclusion $H_i \supset H_{i+1}$. Thus, a decreasing series $L \supset H_1 \supset H_2 \supset \ldots$ of convex normal subloops of loops L is constructed. The relationship $H_{i+1} \supset [H_i, L]$ shows, that this is a decreasing central series. From a finite number of generators of a loop L and the fact that all factors of the chain are torsion-free abelian groups, it follows, that the chain in the finite place ends in unity. Since all factors are preordered, it follows that the loop itself is also pre-ordered.

Thus, it is shown that every finitely generated partially ordered nilpotent automorphic loop (corresponding, Moufang loop), without elements of finite order, is pre-ordered. However, by virtue of the local, if every sub-shell with a finite number of generators of a partially ordered loop is pre-ordered, then the loop itself is also pre-orderable. $\square$

REFERENCES

[1] E.P. Shimbareva, *To the theory of partially ordered groups.* Mat. Coll. **20** (1947), *1*, 145–175.

[2] A.I. Maltsev, *On the ordering of groups* (in Russian). Works Mat. Institute of the Academy of Sciences of the USSR **38** (1951), 173–175.

[3] A.H. Rhemtulla, *Right-ordered groups.* Canad. J. Math. **24** (1972), *5*, 891–895.

[4] R.H. Bruck, *A Survey of Binary Systems.* Berlin-Göttingen-Heidelberg, Springer-Verlag, 1958.

[5] A.V. Kovalschi and V.I. Ursu, *Equational theory of nilpotent A-loop.* Algebra Logic **49** (2010), *4*, 479–497.

[6] R.H. Bruck and L.J. Paige, *Loops whose inner mappings are automorphisms.* Ann. of Math. **68** (1956), 308–323.

[7] V.I. Ursu, *On identities of nilpotent Moufang loops.* Rev. Roumaine Math. Pures Appl. **45** (2000), *3*, 537–548.

Received 22 November 2017

"Simion Stoilov" Institute of Mathematics
of the Romanian Academy,
P.O. Box 1-764,
Bucharest, Romania
Tehnical University,
Chişinău, Republica Moldova
Vasile.Ursu@imar.ro