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On completeness as to ¬-expressibility in a 4-valued provability logic

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Abstract

The simplest non-trivial 4-valued extension $GL4$ of the propositional provability logic GL is considered together with the notion of ¬-expressibility of formulas in GL. The necessary and sufficient conditions when a system of formulas of $GL4$ is complete relative to ¬-expressibility are found out which is formulated in terms of classes formulas that are pre-complete relative to ¬-expressibility.

Keywords: provability logic, expressibility of formulas, completeness relative to ¬-expressibility.

1 Introduction

We consider the simplest 4-valued non-trivial extension of the propositional provability logic GL [1]. A light modification of the notion of expressibility of formulas considered by A.V. Kuznetsov [2] is investigated. The results obtained are similar to the well-known result of E. Post concerning Boolean functions [3].

2 Preliminaries

2.1 Propositional provability logic GL

The propositional provability logic GL is based on propositional variables, logical connectives $\&, \vee, \supset, \neg, \Delta$ [1]. Axioms of GL are the usual axioms of the classical propositional logic together with the following $((\Delta(p \supset q)) \supset (\Delta p \supset \Delta q))$, $\Delta p \supset \Delta\Delta p$, $(\Delta(\Delta p \supset p) \supset \Delta p)$. Rules of inference of the logic L are:

1. *substitution rule* allowing to pass from any formula F to the result of substitution in F instead of a variable p of F by any formula G (denoted by $F(p/G)$, or by $F(G)$),

2. *modus ponens rule* which permits to pass from two formulas F and $F \supset G$ to the formula G, and

3. *rule of necessity* admitting to go from formula F to ΔF.

As usual, by the logic of the calculus GL we understand the set of all formulas that can be deduced in the claculus GL and it is closed with respect to its rules of inference, and we call it the propositional provability logic of Gödel-Löb, denoted also by GL. They say logic L_2 is an extension of the logic L_1 if $L_1 \subseteq L_2$ (as sets).

2.2 Diagonalizable algebras

A diagonalizable algebra [4] $\mathfrak{D}$ is a boolean algebra $\mathfrak{A} = (A; \&, \vee, \supset, \neg, \mathbb{0}, \mathbb{1})$ with an additional operation Δ satisfying the relations:

$$\begin{array}{rcl} \Delta(x \supset y) & = & (\Delta x \supset \Delta y), \\ \Delta x & = & \Delta\Delta x, \\ \Delta(\Delta x \Delta x) & = & \Delta x, \\ \Delta\mathbb{1} & = & \mathbb{1}, \end{array}$$

where $\mathbb{1}$ is the unit of $\mathfrak{A}$. The diagonalizable algebras serve as algebraic models for propositional provability logic GL [1].

It is known [5, 6] that diagonalizable algebras can serve as algebraic models for propositional provability logic. Interpreting logical connectives of a formula F by corresponding operations on a diagonalizable algebra $\mathfrak{D}$ we can evaluate any formula of GL on any algebra $\mathfrak{D}$. If for any evaluation of variables of F by elements of $\mathfrak{D}$ the resulting value of the formula F on $\mathfrak{D}$ is $\mathbb{1}$, they say F *is valid on* $\mathfrak{D}$. The set of all valid formulas on the given diagonalizable algebra $\mathfrak{D}$ is an extension of GL [7], denoted by $L\mathfrak{D}$. So, if diagonalizable algebra $\mathfrak{D}_2$ is a subalgebra of the diagonalizable algebra $\mathfrak{D}_1$, then $L\mathfrak{D}_1 \subseteq L\mathfrak{D}_2$, i.e. $L\mathfrak{D}_2$ is an extension of $L\mathfrak{D}_1$.

Consider the 4-valued diagonalizable algebra $\mathfrak{B}_2 = (\{\mathbb{0}, \rho, \sigma, \mathbb{1}\}; \&, \vee, \supset, \neg, \Delta)$, where $\Delta\mathbb{0} = \Delta\rho = \sigma, \Delta\sigma = \Delta\mathbb{1} = \mathbb{1}$ and its corresponding 4-valued provability logic $L\mathfrak{B}_2$.

2.3 ¬-expressibility of formulas

They say formula F of the logic L is ¬-expressible (see for example de definition of expressibility in [2]) via a system of formulas Σ in L if F can be obtained from variables and $\Sigma \cup \{\neg p\}$ using finitely many times any of the two rules: weak rule of substitution allowing to pass from any formulas A and B to the result of substitution of any of them in other one instead of any variable, and rule of replacement by equivalent formula (if formulas A and $A \sim B$ are given, then we have also B).

3 Main result

Theorem 1. *There is a relative simple algorithm which allows to determine whether a system of formulas Σ is complete relative to ¬-expressibility of formulas in the provability logic $L\mathfrak{B}_2$.*

4 Conclusion

We can also examine other types of expressibility for formulas, such as: parametric expressibility, existential expressibility, weak expressibility. The case of explicit expressibility was examined earlier in [8].

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