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SKEW RING EXTENSIONS AND GENERALIZED MONOID RINGS

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Abstract. A D-*structure* on a ring A with identity is a family of self-mappings indexed by the elements of a monoid G and subject to a long list of rather natural conditions. The mappings are used to define a generalization of the monoid algebra $A[G]$. We consider two of the simpler types of D-structure. The first is based on a homomorphism from G to $\mathrm{End}(A)$ and leads to a skew monoid ring. We also explore connections between these D-structures and normalizing and subnormalizing extensions. The second type of D-structure considered is built from an endomorphism of A. We use D-structures of this type to characterize rings which can be graded by a cyclic group of order 2.

1. Introduction

A system called a *D-structure* in [6] and introduced in [5] consists of a ring A with an identity 1, a monoid G with identity e and mappings $\sigma_{x,y}\colon A \to A$ satisfying the following condition for all $x, y, z \in G$ and $a, b \in A$.

Condition (A).

(0) For each $x \in G$ and $a \in R$, we have $\sigma_{x,y}(a) = 0$ for almost all $y \in G$.

(i) Each $\sigma_{x,y}$ is an additive endomorphism.

(ii) $\sigma_{x,y}(ab) = \sum_{z\in G} \sigma_{x,z}(a)\sigma_{z,y}(b)$.

(iii) $\sigma_{xy,z} = \sum_{uv=z} \sigma_{x,u} \circ \sigma_{y,v}$.

(iv$_1$) $\sigma_{x,y}(1) = 0$ if $x \neq y$;

(iv$_2$) $\sigma_{x,x}(1) = 1$;

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(iv$_3$) $\sigma_{e,x}(a) = 0$ if $x \neq e$;
(iv$_4$) $\sigma_{e,e}(a) = a$.

For brevity a D-structure described in the notation of Condition (A) will be referred to as "a D-structure σ" or by cognate phrases.

In [5] a sort of "skew" or "twisted" monoid ring $A\langle\langle G, \sigma\rangle$ associated with A and G was constructed by means of the mappings $\sigma_{x,y}$. (The connection with the structures usually called skew monoid rings will be elucidated in the next section.) The multiplication in $A\langle G, \sigma\rangle$ is given by the rule

$$(a \cdot x)(b \cdot y) = a \sum_{z \in G} \sigma_{x,z}(b) \cdot zy$$

and distributivity. Examples include group rings, skew polynomial rings and the Weyl algebras. There are also connections with gradings of rings.

We shall examine two relatively simple types of D-structures: those defined by a monoid homomorphism from G to the monoid of (ring-) endomorphisms of A and those defined by an endomorphism f of A using the fact that if $\delta(a) = a - f(a)$ for all $a \in A$ then δ is an (f, id)-derivation, i.e. $\delta(ab) = \delta(a)b + f(a)\delta(b)$ for all $a, b \in A$. For the former we establish connections with *normalizing extensions* [3] and *subnormalizing extensions* (also known as *triangular extensions* [7], [11]). We obtain criteria for $\mathbb{Z}_2$-gradability by means of the latter.

2. D-structures, skew monoid rings and normalizing extensions

For a ring R with identity 1 and a monoid G with identity e let

$$F : G \to \mathrm{End}(A)$$

be a monoid homomorphism, where $\mathrm{End}(A)$ is the monoid of ring endomorphisms with respect to composition. We obtain a D-structure σ^F by defining $\sigma^F_{x,y}$ to be $F(x)$ if $x = y$ and the zero map otherwise. (This is easily verified; cf. [5, Example 1].) This is the situation where G *acts on* A by endomorphisms.

PROPOSITION 2.1. *If σ is defined by a monoid homomorphism F, then in $A\langle G, \sigma\rangle$ the multiplication is given by the formula*

$$(a \cdot x)(b \cdot y) = a\sigma(x)(b) \cdot xy.$$

If σ is defined by an endomorphism F then Proposition 2.1 says that $A\langle G, \sigma\rangle$ ($= A\langle G, \sigma^F\rangle$) is the *skew monoid ring of G over A* defined by F in the sense of [1]. In particular when G is the free monoid generated by an element X we get the *skew polynomial ring* $A[X,\sigma]$ whose multiplication is given by

$$Xa = F(a)X$$

and distributivity. Every skew polynomial ring skewed by an endomorphism only (i.e. not involving any kind of derivation) arises in this way. (See, e.g. [4].) For related comments on skew polynomial rings involving some sort of derivation, see [5, Example 2 and Section 6] and [6, Section 3]. It is reasonable to also employ the term *skew polynomial ring* for a monoid ring defined by a monoid homomorphism $F\colon G \mapsto \mathrm{End}(R)$ when G is a free monoid or a free commutative monoid, and we shall do so shortly.

Example 2.2. Let G be the infinite cyclic monoid $\{x^0(=e), x, x^2, \ldots, x^n, \ldots\}$, R a ring with identity and $R[t]$ the usual polynomial ring. We define

$$F\colon G \to \mathrm{End}(R[t])$$

by setting $F(x^n)(p(t)) = p(t^{2^n})$ for each n. Then each $F(x^n)$ is a ring endomorphism and F is a monoid homomorphism. Let

$$\sigma^F_{mn} = \sigma^F_{x^m,x^n} = \begin{cases} F(x^n) & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

In $R[t]\langle G, \sigma^F\rangle$ we have $xt = x1 \cdot tx^0 = 1\sigma^F_{11}(t)xx^0 = t^2x$.

Example 2.3. Similarly, if K is a field of prime characteristic p and G is as in the previous example, let $F\colon G \to \mathrm{End}(K)$ take each x^n to μ^n, where μ is the *Frobenius monomorphism*: $\mu(a) = a^p$ for all $a \in K$. The resulting ring $K\langle G, \sigma^F\rangle$ is the *Frobenius polynomial ring* in x over K in which $xa = a^px$ for all $a \in K$.

Example 2.4. Let $R[t_1, t_2, t_3]$ be the polynomial ring in three commuting indeterminates over a ring R with identity, S_3 the symmetric group of degree 3. Since each permutation of $\{t_1, t_2, t_3\}$ defines an automorphism of $R[t_1, t_2, t_3]$, we get a monoid homomorphism $F\colon S_3 \to \mathrm{End}(R[t_1, t_2, t_3])$ by defining $F(\lambda)(t_i) = t_{\lambda(i)}$ for all $\lambda \in S_3$ and $i = 1, 2, 3$ and $F(\lambda)(a) = a$ for all $\lambda \in S_3$, $a \in R$. In $R[t_1, t_2, t_3]\langle S_3, \sigma^F\rangle$, for instance, if ρ is the cycle $(1, 2, 3)$, we have $(x_1\rho)(x_2\lambda) = x_1x_3\rho\lambda$ for each $\lambda \in S_3$.

For a given ring A, the assignment $(G, F) \mapsto A\langle G, \sigma^F\rangle$ is in a sense natural. We have

Proposition 2.5. *Let A be a ring, G, G' monoids, $F\colon G \to \mathrm{End}(A)$, $F'\colon G' \to \mathrm{End}(A)$ monoid homomorphisms. Every monoid homomorphism $\varphi\colon G \to G'$ such that $F' \circ \varphi = F$ induces a unique ring homomorphism $\psi\colon A\langle G, \sigma^F\rangle \to A\langle G', \sigma^{F'}\rangle$.*

Proof. Since $A\langle G,\sigma^F\rangle$ is a free A-module on G, there is a unique module homomorphism $\psi\colon A\langle G,\sigma^F\rangle \to A\langle G',\sigma^{F'}\rangle$ such that $\psi(ag)=a\varphi(g)$ for all $a\in A$, $g\in G$. But ψ also preserves multiplication: for all $a,b\in A$, $x,y\in G$ we have

$$\begin{aligned}\psi(ax\cdot by)&=\psi((aF(x)(b))\cdot xy)=aF(x)(b)\varphi(xy)=aF(x)(b)\varphi(x)\varphi(y)\\&=aF'(\varphi(x))(b)\varphi(x)\varphi(y)=a\varphi(x)\cdot b\varphi(y)=\psi(ax)\psi(by).\qquad\square\end{aligned}$$

Corollary 2.6. *Every ring $A\langle G,\sigma^F\rangle$ is an A-homomorphic image of a skew polynomial ring over A* (*in the sense that there is a surjective ring homomorphism from the skew polynomial ring to $A\langle G,\sigma^F\rangle$ which fixes the elements of A*). *If G is commutative, the indeterminates involved can be assumed to commute.*

Proof. There is a free monoid L and a surjective monoid homomorphism $\phi : L\to G$. Then $F\circ\phi : L\to \operatorname{End}(A)$ is a monoid homomorphism. By Proposition 2.5 there is a ring homomorphism $\theta\colon A\langle L,\sigma^{F\circ\phi}\rangle \to A\langle G,\sigma^F\rangle$ with $\theta(ax)=a\phi(x)$ for all $x\in L$. Since ϕ is surjective, so is θ. If G is commutative we can replace L by a free *commutative* monoid. $\square$

A *normalizing extension* of a ring R with identity is a ring S with the same identity such that $R\subseteq S$, S is a left and a right R-module and there are elements x_i, $i\in I$ of S such that $S=\sum_{i\in I}Rx_i$ and $Rx_i=x_iR$ for every $i\in I$. See [3] for details. Examples include group- and semigroup-rings and matrix rings (with respect to the set of so-called matrix units over the subring of scalar matrices). It is natural to ask when a ring $A\langle G,\sigma\rangle$ is a normalizing extension of A (i.e. $\{a\cdot e : a\in A\}$) with respect to G (i.e. $\{1\cdot x : x\in G\}$).

Theorem 2.7. (i) *For a D-structure σ we have $\sigma_{x,y}=0$ whenever $x\neq y$ if and only if $\sigma=\sigma^F$ for some monoid homomorphism $F\colon G\to\operatorname{End}(A)$.*

(ii) *If $A\langle G,\sigma\rangle$ is a normalizing extension of A with respect to G, then σ satisfies the conditions of* (i). *The converse is true if and only if $\sigma(x)$ is surjective for each $x\in G$.*

Proof. (i) If $\sigma_{x,y}$ is zero whenever $x\neq y$, then by (i), (ii) and (iv$_2$) of Condition (A), each $\sigma_{x,x}$ is an endomorphism of A. By (iii) of Condition (A), $\sigma_{xy,xy}=\sigma_{x,x}\circ\sigma_{y,y}$ for all x,y so since by (iv$_4$) $\sigma_{e,e}=\mathrm{id}$, the correspondence $x\mapsto\sigma_{x,x}$ defines a monoid homomorphism which determines σ. The converse is clear.

(ii) If $A\langle G,\sigma\rangle$ is a normalizing extension, then for each $a\in A$ there exists, for each $x\in G$, an element a' of A such that

$$(a'\cdot e)(1\cdot x)=a'x=xa=(1\cdot x)(a\cdot e).$$

But

$$(a' \cdot e)(1 \cdot x) = a' \sum_{y \in G} \sigma_{e,y}(1) \cdot yx = a' \cdot x,$$

while

$$(1 \cdot x)(a \cdot e) = 1 \sum_{z \in G} \sigma_{x,z}(a) \cdot ze = \sum_{z \in G} \sigma_{x,z}(a) \cdot z,$$

so equating coefficients (as $A\langle G, \sigma\rangle$ is a *free* left A-module) we get $\sigma_{x,x}(a) = a'$ and $\sigma_{x,z}(a) = 0$ for $z \neq x$. Since a and x are arbitrary, the conditions of (i) are met.

If σ is defined by a homomorphism $F\colon G \to \operatorname{End}(A)$ then for each $x \in G$ we have $xa = F(x)(a)x$ for all $a \in A$ so $xA \subseteq Ax$. If each $F(x)$ is surjective, then for every $a \in A$, $x \in G$ there is a $b \in A$ such that $a = F(x)(b)$ and then $ax = F(x)(b)x = xb$, whence $Ax \subseteq xA$. Thus $Ax = xA$ for all $x \in G$ and $A\langle G, \sigma\rangle$ is a normalizing extension. If some $F(x)$ is not surjective, let a be in $A \backslash \Im(F(x))$. Suppose $ax = xc$ for some $c \in A$. Then $ax = xc = F(x)(c)x$, so, as $A\langle G, \sigma\rangle$ is a free left A-module on G, we must have $a = F(x)(c)$, which is impossible. Hence there is no such c and it follows that $Ax \neq xA$ and thus $A\langle G, \sigma\rangle$ is not a normalizing extension of A. $\square$

The converse in (ii) of Theorem 2.7 does not hold in the following example.

EXAMPLE 2.8. We consider the Frobenius polynomial ring (see Example 2.3) over a non-perfect field K of characteristic p. If $a \in K$ then for ax to be in xK there must be an element $c \in K$ with $ax = xc = c^p x$ and thus $a = c^p$.

There is a different sort of "converse" to (ii) of Theorem 2.7. We can ask the question: "When is a normalizing extension S of a ring R a generalized monoid ring corresponding to a D-structure?". Excluding "accidental isomorphisms," the question only makes sense if a generating set for S over R is a monoid and S is a free left R-module on this generating set. These two conditions actually suffice.

THEOREM 2.9. *Let S be a normalizing extension of R generated by a multiplicative submonoid G such that S, with its internal multiplication, is a free left R-module generated by G. Then S is a skew monoid ring over R with respect to G and hence a generalized monoid ring defined by a D-structure.*

PROOF. For every $r \in R$, $g \in G$ there exists an element $r' \in R$ with $gr = r'g$, and r' is uniquely determined by r and g by the freeness of S. We rename r' as r^g, so that $gr = r^g g$ for all $r \in R$, $g \in G$. For $r, s \in R$, $g \in G$ we have

$$(r + s)^g g = g(r + s) = gr + gs = r^g g + s^g g = (r^g + s^g)g;$$

$$(rs)^g g = g(rs) = (gr)s = (r^g g)s = r^g (gs) = r^g s^g g,$$

so by freeness, $(r+s)^g = r^g + s^g$ and $(rs)^g = r^g s^g$. Let $\varphi_g(r) = r_g$ for all r, g. Then each φ_g is a ring endomorphism of R.

For $g, h \in G$, $r \in R$, we have

$$\varphi_{gh}(r)g = r^{gh} g = ghr = g \cdot hr = g \cdot r^h h = gr^h \cdot h = (r^h)^g g = \varphi_g \circ \varphi_h(r)g,$$

so by freeness $\varphi_{gh}(r) = \varphi_g \circ \varphi_h(r)$ for all r. Also, the common identity of R and S is the identity of G and it commutes with all elements of R, so $\varphi_1(r) = r$ for all $r \in R$. Putting this together, we see that there is a monoid homomorphism

$$\varphi\colon G \to \mathrm{End}(R)\colon g \mapsto \varphi_g.$$

In the resulting skew monoid ring we have $gr = \varphi(g)(r)g = \varphi_g(r)g = r^g g$ for all r, g, and this is the original multiplication of S. $\square$

The relationship between our monoid rings and normalizing extensions is a bit more subtle, however. It can happen that a monoid ring $A\langle G, \sigma\rangle$ is isomorphic to a ring which *is* a normalizing extension of an isomorphic copy of A. For many purposes this is as good as being a normalizing extension.

Example 2.10 [6, Example 2.2]. Let G be a group, $R = \sum_{g\in G} R_g$ a G-graded ring. For $x, y \in G$, $r = \sum_{g\in G} r_g g \in R$, let $\sigma_{x,y}(r) = \sum_{g^{-1}xg=y} r_g$. The resulting D-structure is far from being defined by a homomorphism. In $R\langle G, \sigma\rangle$ we have

$$(r \cdot x)(s \cdot y) = r \sum_{g\in G} \sigma_{x,g}(s) \cdot gy$$

for all $r, s \in R$, $x, y \in G$. Hence for $x, y, w, z \in G$, $a_w \in R_w$, $a_z \in R_z$ we have

$$(a_w \cdot x)(a_z \cdot y) = a_w \sum_{g\in G} \sigma_{x,g}(a_z) \cdot gy = a_w \sum_{g\in G} \sum_{h^{-1}xh=g} (a_z)_h \cdot gy.$$

But $(a_z)_h = a_z$ if $h = z$ and zero otherwise, so the only value of g which makes a contribution is $z^{-1}xz$ and so $(a_w \cdot x)(a_z \cdot y) = a_w a_z \cdot z^{-1}xzy$. Thus $R\langle G, \sigma\rangle$ is the generalized group ring of Năstăsescu [9]. This is clearly not a skew group ring. However, as was pointed out in a review [8] of [9], the function $\varphi\colon R[G] \to R\langle G, \sigma\rangle$ with $\varphi(a_w \cdot x) = a_w \cdot wx$ (for all $x, w \in G, a_w \in R_w$) is an isomorphism. Also the correspondence $r = \sum_{g\in G} r_g \mapsto \sum_{g\in G} r_g \cdot g^{-1}$ defines an injective homomorphism $i : R \to R\langle G, \sigma\rangle$. Moreover, $\varphi(i(R)) = Re \cong R$. Now, calculating in $R\langle G, \sigma\rangle$, for all $w, x, y, z \in G$, $a_w \in R_w$, $a_z \in R_z$, we have

$$\varphi(a_w \cdot x)\varphi(a_z \cdot y) = (a_w \cdot w^{-1}x)(a_z \cdot z^{-1}y)$$

$$= a_w a_z \cdot z^{-1}(w^{-1}x)z(z^{-1}y) = a_w a_z \cdot z^{-1}w^{-1}xy = a_w a_z \cdot (wz)^{-1}xy.$$

But $a_w a_z \in R_{wz}$, so $a_w a_z \cdot (wz)^{-1}xy = \varphi(a_w a_z \cdot xy)$. Thus $R\langle G, \sigma\rangle = \varphi(R)[G]$ (by a slight *abus de langage*). The main point is that $R\langle G, \sigma\rangle$, despite being defined by a rather intricate D-structure, is nevertheless a normalizing extension of an isomorphic copy of R.

Our second example of a "disguised skew monoid ring" is not actually a normalizing extension.

Example 2.11. Let $R[x]$ be a polynomial ring, G the free monoid on a single generator y. Writing σ_{mn} instead of σ_{y^m,y^n} we get a D-structure σ from $R[x]$ and G where $\sigma_{00} = \mathrm{id}$, $\sigma_{m,2^jm}(r_0 + r_1x + \cdots + r_kx^k) = r_j$ and all other $\sigma_{mn} = 0$. In $R[x]\langle G, \sigma\rangle$ we have $yx = xy^2$. (This is a special case of [6, Example 2.5].) Thus $R\langle G, \sigma\rangle$ is the ring of polynomials in x and y over R in which the indeterminates commute with the elements of R but $yx = xy^2$. This is not (defined as) a skew polynomial ring and σ is not defined by a homomorphism. But in Example 2.2, renaming G as H and $R[t]$ as $R[y]$, we get the skew polynomial ring $R[y]\langle H, \sigma^F\rangle$, in which $xy = y^2x$. This ring is clearly isomorphic to the opposite ring of $R\langle G, \sigma\rangle$, so that with respect to many important properties it is the same as $R\langle G, \sigma\rangle$, though the latter is defined by a D-structure which is not defined by a monoid homomorphism. Note that $R[y]\langle H, \sigma^F\rangle$ is not a normalizing extension as its mappings are not surjective (Theorem 2.7), but in this case as in the previous example a generalized monoid ring is produced whose relation to the coefficient ring cannot be immediately deduced from the form of the D-structure which defines it.

3. Subnormalizing extensions

A ring S with identity is a *subnormalizing extension* or a *triangular extension* of a subring R with the same identity with respect to a finite or countably infinite set of elements x_n (n being a positive integer) if $S = \sum Rx_n = \sum x_nR$ and for each (relevant) j we have $\sum_{n\le j} Rx_j = \sum_{n\le j} x_nR$. It is usual to take x_1 to be the common identity of R and S. Normalizing extensions are subnormalizing extensions but the converse is false in general. For further details see [7], [11]. For the monoid ring corresponding to a D-structure to be a subnormalizing extension with respect to its monoid it is not necessary that the D-structure be defined by a monoid homomorphism.

Example 3.1. Let A be a ring with identity, $G = \{e, x\}$ a cyclic group of order 2 (so that $x^2 = e$). Let δ be a derivation of A such that $2\delta = 0 = \delta^2$. For an example of such a derivation, let c be a non-central element of A such that $2c = 0 = c^2$ and $\delta(a) = [c, a]$ for all $a \in A$. We get a D-structure for A and G by defining

$$\sigma_{ee} = \mathrm{id} = \sigma_{xx}; \quad \sigma_{ex} = 0; \quad \sigma_{xe} = \delta.$$

Everything is pretty straightforward. Here are two parts of the verification of Condition (A)(iii), which are the only occasions where the assumptions about δ are needed. (Alternatively, all parts of Condition (A) can be deduced from Examples 4.3 and 4.4 below.) We have

$$\sigma_{xx} \circ \sigma_{xx} + \sigma_{xe} \circ \sigma_{xe} = \mathrm{id}^2 + \delta^2 = \mathrm{id} = \sigma_{ee} = \sigma_{xx,e}$$

and

$$\sigma_{xe} \circ \sigma_{xx} + \sigma_{xx} \circ \sigma_{xe} = \delta \circ \mathrm{id} + \mathrm{id} \circ \delta = 2\delta = 0 = \sigma_{ex} = \sigma_{xx,x}.$$

Since $\sigma_{xe} \neq 0$, the corresponding monoid ring is not a skew monoid ring. Now for each $a \in A$, we have $xa = (1 \cdot e)(a \cdot e) = \sigma_{xe}(a) \cdot ee + \sigma_{xx}(a) \cdot xe = \delta(a)e + ax$. Thus

$$ea = ae, \quad xa = ax + \delta(a)e \in Ax + Ae$$

and

$$ax = xa - \delta(a)e \in xA + eA,$$

so that $A\langle G, \sigma\rangle$ is a subnormalizing extension of A with respect to $\{e, x\}$.

If we take $H = \{e, x\}$ with $x^2 = x$ but now require that $2\delta = 0$ and $\delta^2 = \delta$ (e.g. if δ is the inner derivation defined by a non-central idempotent u with $2u = 0$) we get a D-structure for A and H using the same mappings as for G. Again the resulting ring $A\langle H, \sigma\rangle$ is a subnormalizing extension but not a skew monoid ring. (Apart from the verification of Condition (A)(iii) everything is the same as in the example using G.)

Example 3.2. The (first) Weyl algebra over a field K can be viewed as the monoid ring over the ordinary polynomial ring $K[x]$ defined by an infinite cyclic monoid $\langle y\rangle$ with respect to a D-structure. (See [5, Example 2.1].) It is clear from the specification of this D-structure that the algebra is not a skew monoid ring. It is, however, a subnormalizing extension with respect to $\{y^n : n \geq 0\}$. This is shown as follows. We have $yx = xy + 1$, from which it follows easily by induction that $y^n x = xy^n + ny^{n-1}$ for all $n \geq 1$. For each $n \geq 1$, a further induction (on m) shows that $y^n x^m \in \sum_{i=0}^{n} K[x]y^i$ for each m. Similarly, starting from the relation $xy = yx - 1$ we can show that $x^m y^n \in \sum_{i=0}^{n} y^i K[x]$.

Example 3.3. Consider now the third Weyl algebra. Again as in [5, Example 2.1] we see this as a monoid algebra over a polynomial ring $K[x_1, x_2, x_3]$ determined by the free commutative monoid generated by $\{y_1, y_2, y_3\}$. The elements of this monoid, the "monic monomials", can be labelled by their ordered triples of indices, and the triples given the lexicographic order, which is linear: $(r_1, r_2, r_3) < (s_1, s_2, s_3)$ when at the first place where they differ (reading from left to right), say the tth place, we have $r_t < s_t$. Then the lowest label, $(0,0,0)$, goes with the identity. Every monomial in $K[x_1, x_2, x_3]$ has the form $ax_1^{m_1}x_2^{m_2}x_3^{m_3}$, where $a \in K$, since the xs commute. Then as the ys commute, every monomial in the Weyl algebra has the form $\alpha y_1^{n_1}y_2^{n_2}y_3^{n_3}$, where $\alpha \in K[x_1, x_2, x_3]$. It may be assumed that α is a monomial; let $\alpha = ax_1^{m_1}x_2^{m_2}x_3^{m_3}$. Now

$$\alpha y_1^{n_1}y_2^{n_2}y_3^{n_3} = ax_1^{m_1}x_2^{m_2}x_3^{m_3}y_1^{n_1}y_2^{n_2}y_3^{n_3} = ax_1^{m_1}x_2^{m_2}y_1^{n_1}y_2^{n_2}(x_3^{m_3}y_3^{n_3})$$

(as x_3 commutes with y_1 and y_2)

$$= a(x_1^{m_1}y_1^{n_1})(x_2^{m_2}y_2^{n_2})(x_3^{m_3}y_3^{n_3})$$

(by similar arguments). Now each x and the corresponding y generate an isomorphic copy of the first Weyl algebra over K, so by what was proved about that algebra above there are polynomials $\alpha_i(x_1)$ in x_1, $\beta_j(x_2)$ in x_2 and $\gamma_\ell(x_3)$ in x_3 such that

$$\alpha y_1^{n_1}y_2^{n_2}y_3^{n_3} = a\sum_{i=1}^{n_1} y_1^i\alpha_i(x_1)\sum_{j=1}^{n_2} y_2^j\beta_j(x_2)\sum_{\ell=1}^{n_3} y_3^\ell\gamma_\ell(x_3)$$

$$= a\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{\ell=1}^{n_3} y_1^i y_2^j y_3^\ell \alpha_i(x_1)\beta_j(x_2)\gamma_\ell(x_3),$$

as x_2 commutes with y_3 and x_1 commutes with y_2 and y_3. Note that as $i \leq n_1$, $j \leq n_2$ and $\ell \leq n_3$ we have $(i, j, \ell) \leq (n_1, n_2, n_3)$ lexicographically. Thus the left module generated by an element of the monoid is contained in the right module generated by itself and "earlier" elements. The corresponding result with "left" and "right" interchanged follows from symmetry as in the case of the first Weyl algebra above. This proves that the third Weyl algebra is a subnormalizing extension.

In the case of characteristic p, it is shown by a different method in [7, Example 4], that the Weyl algebra of degree $2r$ is a subnormalizing extension of a polynomial ring.

4. More about subnormalizing extensions

Let S be a subnormalizing extension of R with respect to $\{x_1, x_2, \dots, x_n\}$ or $\{x_1, x_2, \dots, x_n, \dots\}$, where $x_1 = 1 =$ the identity of R. In view of Theorem 2.9 it is reasonable to ask whether there is something like a D-structure concealed in these specifications. If our extension is defined by a D-structure then *S is a free R-module with the x_i as a basis and this basis is a monoid with respect to the multiplication of S, with identity* 1. We therefore assume that these conditions are met. Then $x_1 r = r = r x_1$ for each $r \in R$. For such r, as $x_2 r \in x_1 R + x_2 R = R x_1 + R x_2$, there are *unique* elements r_{21}, r_{22} of R such that $x_2 r = r_{21} x_1 + r_{22} x_2$. Similarly $x_3 r = r_{31} x_1 + r_{32} x_2 + r_{33} x_3$ with unique coefficients, and in general for $i \leq n$ we have $x_i r = \sum_{j \leq i} r_{ij} x_j$. For uniformity let $r_{11} = r$ for each r.

We define functions $\sigma_{x_i,x_j} \colon R \to R$ for $1 \leq j \leq i \leq n$ (where n is the number of elements in the monoid) by setting $\sigma_{x_i,x_j}(r) = r_{ij}$ for all r. For $1 \leq i < j \leq n$ let σ_{x_i,x_j} be the zero function. It will sometimes be convenient to write $x_i r = \sum_{j=i}^{n} \sigma_{x_i,x_j}(r) x_j$. Then for any i and all $r, t \in R$ we have

$$x_i(r+t) = x_i r + x_i t = \sum_{j=1}^{n} \sigma_{x_i,x_j}(r) x_j + \sum_{j=1}^{n} \sigma_{x_i,x_j}(t) x_j$$

$$= \sum_{j=1}^{n} (\sigma_{x_i,x_j}(r) + \sigma_{x_i,x_j}(t)) x_j$$

and

$$x_i(rt) = (x_i r)t = \left(\sum_{j=1}^{n} \sigma_{x_i,x_j}(r) x_j \right) t = \sum_{j=1}^{n} \sigma_{x_i,x_j}(r)(x_j t)$$

$$= \sum_{j=1}^{n} \sigma_{x_i,x_j}(r) \left(\sum_{k=1}^{n} \sigma_{x_j,x_k}(t) x_k \right) = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} \sigma_{x_i,x_j}(r) \sigma_{x_j,x_k}(t) \right) x_k.$$

In this expression the coefficient of x_k is $\sum_{j=1}^{n} \sigma_{x_i,x_j}(r) \sigma_{x_j,x_k}(t)$. Thus

$$\sigma_{x_i,x_j}(r+t) = \sigma_{x_i,x_j}(r) + \sigma_{x_i,x_j}(t); \sigma_{x_i,x_j}(rt) = \sum_{k=1}^{n} \sigma_{x_i,x_k}(r) \sigma_{x_k,x_j}(t)$$

for all $r, t \in R$, $1 \leq i \leq n$, $1 \leq j \leq n$ (where we have interchanged j and k for uniformity). Whenever it is more convenient we shall use the more compact version:

$$\sigma_{x_i,x_j}(rt) = \sum_{k=j}^{i} \sigma_{x_i,x_k}(r) \sigma_{x_k,x_j}(t) \quad \text{for } i \geq j.$$

Hence our functions satisfy (i) and (ii) of Condition (A). Since for every i we have $x_i 1 = 1 x_i = 1 x_i + \sum_{1 < j \leq i} 0 x_j$ we have $\sigma_{ii}(1) = 1$ and $\sigma_{x_i,x_j}(1) = 0$ for $j \neq i$, i.e. (iv_1) and (iv_2) of Condition (A) are satisfied. If $i \neq 1$ then $i > 1$ so σ_{1i} is the zero map, while $\sigma_{11}(r) = r_{11} = r$ for all $r \in R$ so we have (iv_3) and (iv_4) too.

Part (iii) of Condition (A) is more complicated; it remains an open question whether it has to be satisfied or not. It is used in [5] only to prove that the rings $A\langle G,\sigma\rangle$ are associative (pp. 35–36), and it is not known whether or not it is *necessary* for associativity. On the other hand, subnormalizing extensions are certainly associative, so there are two possibilities: either

(1) a weaker condition than (iii) suffices for associativity or

(2) all subnormalizing extensions (satisfying the obvious requirements) come from D-structures.

Be all this as it may, there *are* monoids M such that every subnormalizing extension defined by the adjunction of the elements of M which is a free left module over the ground ring defines a D-structure. We will give examples shortly, but first we will show that when a D-structure does result, its corresponding monoid ring coincides with the subnormalizing extension.

THEOREM 4.1. *Let $G=\{x_1,x_2,\dots,x_n\}$ or $\{x_1,x_2,\dots,x_n,\dots\}$ be a monoid with identity x_1, $S=\sum Rx_i$ a subnormalizing extension of a ring R in which the monoid multiplication agrees with the ring multiplication on the x_i and which is a free left R-module in which the x_i constitute a basis. For $i\geq j$ and for $r\in R$ let $\sigma_{x_i,x_j}(r)=r_{ij}$, where $x_ir=r_{i1}x_1+r_{i2}x_2+\dots+r_{ii}x_i$. For $i<j$ let σ_{x_i,x_j} be the zero map. If the mappings σ_{x_i,x_j} form a D-structure σ, then $R\langle G,\sigma\rangle=S$.*

PROOF. We make the appropriate identifications between elements of G and the corresponding elements of $R\langle G,\sigma\rangle$. Then calculating in $R\langle G,\sigma\rangle$ we get, for $r,s\in R$ and all i, j,

$$(rx_i)(sx_j)=r\sum_{k=1}^{i}\sigma_{x_i,x_k}(s)x_kx_j=r\left(\sum_{k=1}^{i}s_{ik}x_k\right)x_j=r(x_is)x_j=rx_isx_j,$$

and the last term represents the product of the two elements in S. □

Suppose, in the notation of the theorem, that S is a normalizing extension of R with respect to G. Then for each i and for each $r\in R$ there is an $r_i\in R$ for which $x_ir=r_ix_i$. This means that $\sigma_{x_i,x_i}(r)=r_i$ and $\sigma_{x_i,x_j}(r)=0$ when $i\neq j$ as in Theorem 2.9.

As we mentioned before, there are monoids for which all subnormalizing extensions as in the theorem define D-structures. We now present two of these. All the notation of the theorem and the preceding discussion will be used without further comment.

EXAMPLE 4.2. Let $G=\{x_1,x_2\}$ be a cyclic group of order 2 with identity x_1. In verifying (iii) of Condition (A) (which is all we have to do) we need all representation of the elements of G as products. They are

$$x_1=x_1x_1=x_2x_2;\quad x_2=x_1x_2=x_2x_1.$$

Now

$$\sigma_{x_1x_2,x_1} = \sigma_{x_2,x_1}(r) = r_{21},$$

while

$$\sigma_{x_1,x_1} \circ \sigma_{x_2,x_1}(r) + \sigma_{x_1,x_2} \circ \sigma_{x_2,x_2}(r) = (r_{21})_{11} + 0 = r_{21}.$$
$$\sigma_{x_1x_2,x_2}(r) = \sigma_{x_2,x_2}(r) = r_{22};$$
$$\sigma_{x_1,x_1} \circ \sigma_{x_2,x_2}(r) + \sigma_{x_1,x_2} \circ \sigma_{x_2,x_1}(r) = (r_{22})_{11} + 0 = r_{22}.$$
$$\sigma_{x_2x_1,x_1}(r) = \sigma_{x_2,x_1}(r) = r_{21};$$
$$\sigma_{x_2,x_1} \circ \sigma_{x_1,x_1}(r) + \sigma_{x_2,x_2} \circ \sigma_{x_1,x_2}(r) = (r_{11})_{21} + 0 = r_{21}.$$
$$\sigma_{x_2x_1,x_2}(r) = \sigma_{x_2,x_2}(r) = r_{22};$$
$$\sigma_{x_2,x_1} \circ \sigma_{x_1,x_2}(r) + \sigma_{x_2,x_2} \circ \sigma_{x_1,x_1}(r) = 0 + (r_{11})_{22} = r_{22}.$$
$$\sigma_{x_1x_1,x_1}(r) = \sigma_{x_1,x_1}(r) = r_{11} = r;$$
$$\sigma_{x_1,x_1} \circ \sigma_{x_1,x_1}(r) + \sigma_{x_1,x_2} \circ \sigma_{x_1,x_2}(r) = (r_{11})_{11} + 0 = r.$$
$$\sigma_{x_1x_1,x_2}(r) = \sigma_{x_1,x_2}(r) = 0; \quad \sigma_{x_1,x_1} \circ \sigma_{x_1,x_2}(r) + \sigma_{x_1,x_2} \circ \sigma_{x_1,x_1}(r) = 0.$$

Before treating the last two cases we do a helpful calculation. We have

$$\begin{aligned} x_1r = (x_2x_2)r = x_2(x_2r) &= x_2(r_{21}x_1 + r_{22}x_2) = (x_2r_{21})x_1 + (x_2r_{22})x_2 \\ &= (r_{21})_{21}x_1x_1 + (r_{21})_{22}x_2x_1 + (r_{22})_{21}x_1x_2 + (r_{22})_{22}x_2x_2 \\ &= (r_{21})_{21}x_1 + (r_{21})_{22}x_2 + (r_{22})_{21}x_2 + (r_{22})_{22}x_1. \end{aligned}$$

But $x_1r = rx_1 + 0x_2$, so equating coefficients we get

(1) $$(r_{21})_{21} + (r_{22})_{22} = r; \quad (r_{21})_{22} + (r_{22})_{21} = 0.$$

We now treat the remaining cases.

$$\sigma_{x_2x_2,x_1}(r) = \sigma_{x_1,x_1}(r) = r_{11} = r;$$
$$\sigma_{x_2,x_1} \circ \sigma_{x_2,x_1}(r) + \sigma_{x_2,x_2} \circ \sigma_{x_2,x_2}(r) = (r_{21})_{21} + (r_{22})_{22} = r$$

by (1).

$$\sigma_{x_2x_2,x_2}(r) = \sigma_{x_1,x_2}(r) = 0;$$
$$\sigma_{x_2,x_1} \circ \sigma_{x_2,x_2}(r) + \sigma_{x_2,x_2} \circ \sigma_{x_2,x_1}(r) = (r_{22})_{21} + (r_{21})_{22} = 0$$

by (1).

We note that the cyclic group of order 3 does not have this property.

EXAMPLE 4.3. The two element semilattice with identity also provides "automatic D-structures". Let $G = \{x_1, x_2\}$ where ${x_2}^2 = x_2$ and x_1 is an identity. We shall show that every subnormalizing extension defined by G defines a D-structure as in the theorem. We begin by obtaining some information which will help with the verification of (iii) of Condition (A).

For each $r \in R$ we have

$$\begin{aligned} r_{21}x_1 + r_{22}x_2 = x_2 r = x_2(x_2 r) = x_2(r_{21}x_1 + r_{22}x_2) &= (x_2 r_{21})x_1 + (x_2 r_{22})x_2 \\ &= ((r_{21})_{21}x_1 + (r_{21})_{22}x_2)x_1 + ((r_{22})_{21}x_1 + (r_{22})_{22}x_2)x_2 \\ &= (r_{21})_{21}x_1 + [(r_{21})_{22} + (r_{22})_{21} + (r_{22})_{22}]x_2, \end{aligned}$$

so

$$(r_{21})_{21} = r_{21}; \quad (r_{21})_{22} + (r_{22})_{21} + (r_{22})_{22} = r_{22}. \tag{2}$$

Now we check the various cases in (iii) of Condition (A).

$$\sigma_{x_1x_1,x_1}(r) = \sigma_{x_1,x_1}(r) = r_{11} = r; \quad \sigma_{x_1,x_1} \circ \sigma_{x_1,x_1}(r) = (r_{11})_{11} = r.$$

$$\sigma_{x_1x_1,x_2}(r) = \sigma_{x_1,x_2}(r) = 0;$$

$$\sigma_{x_1,x_1} \circ \sigma_{x_1,x_2}(r) + \sigma_{x_1,x_2} \circ \sigma_{x_1,x_1}(r) + \sigma_{x_1,x_2} \circ \sigma_{x_1,x_2}(r) = 0 + 0 + 0 = 0.$$

$$\sigma_{x_1x_2,x_1}(r) = \sigma_{x_2,x_1}(r) = r_{21}; \quad \sigma_{x_1,x_1} \circ \sigma_{x_2,x_1}(r) = (r_{21})_{11} = r_{21}.$$

$$\sigma_{x_1x_2,x_2}(r) = \sigma_{x_2,x_2}(r) = r_{22};$$

$$\begin{aligned} &\sigma_{x_1,x_1} \circ \sigma_{x_2,x_2}(r) + \sigma_{x_1,x_2} \circ \sigma_{x_2,x_1}(r) + \sigma_{x_1,x_2} \circ \sigma_{x_2,x_2}(r) \\ &= (r_{22})_{11} + 0 + 0 = r_{22}. \end{aligned}$$

$$\sigma_{x_2x_1,x_1}(r) = \sigma_{x_2,x_1}(r) = r_{21}; \quad \sigma_{x_2,x_1} \circ \sigma_{x_1,x_1}(r) = (r_{11})_{21} = r_{21}.$$

$$\sigma_{x_2x_1,x_2}(r) = \sigma_{x_2,x_2}(r) = r_{22};$$

$$\begin{aligned} &\sigma_{x_2,x_1} \circ \sigma_{x_1,x_2}(r) + \sigma_{x_2,x_2} \circ \sigma_{x_1,x_1}(r) + \sigma_{x_2,x_2} \circ \sigma_{x_1,x_2}(r) \\ &= 0 + (r_{11})_{22} + 0 = r_{22}. \end{aligned}$$

$$\sigma_{x_2x_2,x_1}(r) = \sigma_{x_2,x_1}(r) = r_{21}; \quad \sigma_{x_2,x_1} \circ \sigma_{x_2,x_1}(r) = (r_{21})_{21} = r_{21}$$

by (2).

$$\sigma_{x_2x_2,x_2}(r) = \sigma_{x_2,x_2}(r) = r_{22};$$

$$\begin{aligned} &\sigma_{x_2,x_1} \circ \sigma_{x_2,x_2}(r) + \sigma_{x_2,x_2} \circ \sigma_{x_2,x_1}(r) + \sigma_{x_2,x_2} \circ \sigma_{x_2,x_2}(r) \\ &= (r_{22})_{21} + (r_{21})_{22} + (r_{22})_{22} = r_{22} \end{aligned}$$

by (2).

Since we have agreement in all cases, (iii) of Condition (A) is satisfied and we have a D-structure.

Now here are some examples to further illustrate the relationship between subnormalizing extensions and D-structures over the monoids of Examples 4.3 and 4.4. We begin with a subnormalizing extension defined by a monoid which does not give rise to a D-structure.

Example 4.4. Let K be a field,

$$R=\left\{\begin{bmatrix}a&b\\0&a\end{bmatrix}: a,b\in K\right\},\quad S=\begin{bmatrix}K&K\\0&K\end{bmatrix},\quad x_1=\begin{bmatrix}1&0\\0&1\end{bmatrix}\quad\text{and}\quad x_2=\begin{bmatrix}1&0\\0&0\end{bmatrix}.$$

For all $a,b\in K$ we have

$$\begin{bmatrix}a&b\\0&a\end{bmatrix}x_2=\begin{bmatrix}a&0\\0&0\end{bmatrix}=x_2\begin{bmatrix}a&0\\0&a\end{bmatrix}$$

and

$$x_2\begin{bmatrix}a&b\\0&a\end{bmatrix}=\begin{bmatrix}a&b\\0&0\end{bmatrix}=\begin{bmatrix}a&0\\0&a\end{bmatrix}x_2+\begin{bmatrix}0&b\\0&0\end{bmatrix}x_1.$$

Hence S is a subnormalizing extension defined by $\{x_1,x_2\}$ which is a monoid with identity x_1 and $x_2={x_2}^2$. We do not have a D-structure as S is not a free left R-module on $\{x_1,x_2\}$. (One way to see this is by observing K-dimension: R has dimension 2 so if S were a free module on two generators it would have dimension 4; its dimension is 3, however.)

Example 4.5. Examples 4.2 and 4.3 provide a quicker way to verify that the D-structures of Example 3.1 really *are* D-structures. The proof, given in Examples 4.2 and 4.3, that we have subnormalizing extensions, is much quicker.

Our next example gives a D-structure defined by a cyclic group of order 2 for which the monoid ring is not a subnormalizing extension.

Example 4.6. Let $G=\{e,x\}$ where e is the identity and $x^2=e$. On the ring $\left[\begin{smallmatrix}K&K\\0&K\end{smallmatrix}\right]$ let

$$\sigma_{x,x}\left(\begin{bmatrix}a&b\\0&c\end{bmatrix}\right)=\begin{bmatrix}a&0\\0&a\end{bmatrix},\ \sigma_{x,e}\left(\begin{bmatrix}a&b\\0&c\end{bmatrix}\right)=\begin{bmatrix}0&c-a\\0&b\end{bmatrix},\ \sigma_{e,x}=0,\ \sigma_{e,e}=\mathrm{id}.$$

This gives us a D-structure σ. But in $\left[\begin{smallmatrix}K&K\\0&K\end{smallmatrix}\right]\langle G,\sigma\rangle$ we have, for all $a,b,c\in K$,

$$x\begin{bmatrix}a&b\\0&c\end{bmatrix}=\sigma_{xx}\left(\begin{bmatrix}a&b\\0&c\end{bmatrix}\right)xe+\sigma_{xe}\left(\begin{bmatrix}a&b\\0&c\end{bmatrix}\right)ee=\begin{bmatrix}a&0\\0&a\end{bmatrix}x+\begin{bmatrix}0&c-a\\0&b\end{bmatrix}e.$$

Suppose for some $r, s, t, p, q, \ell \in K$ we have

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} x = x \begin{bmatrix} r & s \\ 0 & t \end{bmatrix} + e \begin{bmatrix} p & q \\ 0 & \ell \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} x + \begin{bmatrix} 0 & t-r \\ 0 & s \end{bmatrix} e + \begin{bmatrix} p & q \\ 0 & \ell \end{bmatrix} e.$$

Then $b = 0$ and $a = c$. Hence $\left[\begin{smallmatrix} K & K \\ 0 & K \end{smallmatrix}\right] \langle G, \sigma \rangle$ is not a subnormalizing extension (not with respect to G anyway; cf. Section 2.)

EXAMPLE 4.7. We get a similar example to the preceding one with $x^2 = x$ instead of the cyclic group by taking

$$\sigma_{x,x}\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \ \sigma_{x,e}\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & b+c-a \end{bmatrix}, \ \sigma_{e,x} = 0, \ \sigma_{e,e} = \mathrm{id}.$$

5. Rings graded by a cyclic group of order 2

In this section we shall obtain a characterization in terms of D-structures of rings which can be graded by the cyclic group $\mathbb{Z}_2$ of order 2.

THEOREM 5.1. *Consider the following conditions on a ring A with identity.*

(i) *A is $\mathbb{Z}_2$-graded.*

(ii) *A has an automorphism f with $f^2 = \mathrm{id}$ and an idempotent (f, id)-derivation δ such that $f\delta = \delta f = -\delta$.*

(iii) *A has an automorphism f with $f^2 = \mathrm{id}$ and an idempotent (f, id)-derivation δ such that $a = f(a) + 2\delta(a)$ for all $a \in A$.*

(iv) *A has an automorphism f with $f^2 = \mathrm{id}$ such that $a - f(a) \in 2A$ for all $a \in A$. Conditions* (i), (ii) *and* (iii) *are equivalent and they imply* (iv). *If A is 2-torsion-free, then all four conditions are equivalent.*

PROOF. (i) $\Rightarrow$ (ii). We take $\mathbb{Z}_2 = \{0, 1\}$ and let A_0, A_1 be the components corresponding to $0, 1$ respectively. Thus $A = A_0 + A_1$ and each element of A has a unique representation $a = a_0 + a_1$, $a_0 \in A_0$, $a_1 \in A_1$. Let $f(a) = f(a_0 + a_1) = a_0 - a_1$ for each $a \in A$. If $b = b_0 + b_1 \in A$, then

$$ab = (a_0b_0 + a_1b_1) + (a_0b_1 + a_1b_0),$$

where the bracketed terms are in A_0, A_1 respectively. Hence

$$\begin{aligned} f(a)f(b) &= (a_0 - a_1)(b_0 - b_1) = a_0b_0 - a_0b_1 - a_1b_0 + a_1b_1 \\ &= (a_0b_0 + a_1b_1) - (a_0b_1 + a_1b_0) = f(ab). \end{aligned}$$

Since clearly f preserves addition and $f^2 = \mathrm{id}$, f meets all the requirements of (ii). Let $\delta(a) = a_1$ for all $a \in A$. Then for all $a, b \in A$ we have

$$\delta(a)b + f(a)\delta(b) = a_1b + (a_0 - a_1)b_1$$

$$= a_1(b_0 + b_1) + a_0b_1 - a_1b_1 = a_1b_0 + a_0b_1 = \delta(ab).$$

Thus δ is an (f, id)-derivation and clearly idempotent. For all $a \in A$ we have $f\delta(a) = f(a_1) = -a_1 = -\delta(a)$ and $\delta f(a) = \delta(a_0 - a_1) = -a_1 = -\delta(a)$.

(ii) $\Rightarrow$ (i). Let $A_0 = \{a \in A : \delta(a) = 0\}$ and $A_1 = \{a \in A : \delta(a) = a\}$. Then $A_0 \cap A_1 = 0$ and for every $a \in A$ we have $a = (a - \delta(a)) + \delta(a)$, where $\delta(a - \delta(a)) = \delta(a) - \delta^2(a) = 0$ and $\delta(\delta(a)) = \delta(a)$. Thus additively $A = A_0 \oplus A_1$. If $a, b \in A_0$, then $\delta(ab) = \delta(a)b + f(a)\delta(b) = 0$, so $ab \in A_0$. If $a, b \in A_1$, then

$$\delta(ab) = \delta^2(ab) = \delta(\delta(a)b + f(a)\delta(b))$$
$$= \delta^2(a)b + f\delta(a)\delta(b) + \delta f(a)\delta(b) + f^2(a)\delta^2(b)$$
$$= \delta(a)b - \delta(a)\delta(b) - \delta(a)\delta(b) + a\delta(b) = ab - ab - ab + ab = 0,$$

so $ab \in A_0$. If $a \in A_0$ and $b \in A_1$, then

$$\delta(ab) = \delta^2(ab) = \delta(a)b - \delta(a)\delta(b) - \delta(a)\delta(b) + a\delta(b) \quad \text{(as above)}$$
$$= 0 - 0 - 0 + ab = ab,$$

so $ab \in A_1$. Also $\delta(ba) = \delta(b)a + f(b)\delta(a) = ba + 0 = ba$, so $ba \in A_1$. Thus $A_0A_0, A_1A_1 \subseteq A_0$ and $A_0A_1, A_1A_0 \subseteq A_1$, so A_0 and A_1 are the components of a $\mathbb{Z}_2$-grading of A.

(i) $\Rightarrow$ (iii). We preserve the relevant notation from (i) $\Rightarrow$ (ii). Then f as defined there is an automorphism with $f^2 = \mathrm{id}$. Let $\delta(a) = \delta(a_0 + a_1 = a_1)$ for every $a \in A$. As in (i) $\Rightarrow$ (ii), δ is a (f, id)-derivation. Now for every a we have $a - f(a) = a_0 + a_1 - (a_0 - a_1) = 2a_1 \in 2A$.

(iii) $\Rightarrow$ (i). Since δ is idempotent, we have $A = A_0 \oplus A_1$ additively as in (ii) $\Rightarrow$ (i). If $a, b \in A_0$, then $\delta(ab) = \delta(a)b + f(a)\delta(b) = 0$, so $ab \in A_0$. If $a, b \in A_1$, then $\delta(ab) = \delta(a)b + f(a)\delta(b) = ab + f(a)b$. But $a \in A_1$, so $2a = 2\delta(a) = a - f(a)$, whence $f(a) = -a$ and so $\delta(ab) = ab - ab = 0$, i.e. $ab \in A_0$. If $a \in A_0$ and $b \in A_1$, then $a - f(a) = 2\delta(a) = 0$, so $f(a) = a$. Now $\delta(ab) = \delta(a)b + f(a)\delta(b) = 0b + ab = ab$, so $ab \in A_1$. Finally, $\delta(ba) = \delta(b)a + f(b)\delta(a) = ba + f(b)0 = ba$ so ba is also in A_1. This proves that A is $\mathbb{Z}_2$-graded with components A_0, A_1.

Clearly (iii) $\Rightarrow$ (iv).

If A satisfies (iv), let $d(a) = a - f(a)$ for all $a \in A$. Then for all $a, b \in A$ we have

$$f(a)d(b) + d(a)b = f(a)(b - f(b)) + (a - f(a))b$$
$$= f(a)b - f(a)f(b) + ab - f(a)b = ab - f(ab) = d(ab),$$

so d is an (f, id)-derivation. If A is 2-torsion-free (and satisfies (iv)), then for each a, there is a *unique* $x \in A$ such that $d(a) = 2x$. Re-naming x as $\frac{1}{2}d(a)$

we implicitly define $\frac{1}{2}d$. Since $d(ab) = 2(f(a)\cdot\frac{1}{2}d(b)+\frac{1}{2}d(a)b)$, by uniqueness we have

$$\tfrac{1}{2}d(ab) = f(a)\cdot\tfrac{1}{2}d(b) + \tfrac{1}{2}d(a)\cdot b$$

so $\frac{1}{2}d$ is also an (f,id)-derivation. Also $d^2(a) = a - f(a) - f(a - f(a)) = 2d(a)$ for all a, since $f^2 = f$, so $4(\frac{1}{2}d)^2(a) = (2\frac{1}{2}d)^2(a) = d^2(a) = 2d(a) = 4\cdot\frac{1}{2}d(a)$, so again since A is 2-torsion-free, we have $(\frac{1}{2}d)^2 = \frac{1}{2}d$. Since $a = f(a) + d(a) = f(a) + 2\cdot\frac{1}{2}d(a)$ for all a, A satisfies (iii) and the proof is now complete. □

If A satisfies (i), (ii) and (iii) of the theorem and $2A = 0$, then in (iii) $f = \mathrm{id}$ so δ is an ordinary derivation. If, on the other hand, A is 2-torsion-free, then f determines δ ($\delta(a) = \frac{1}{2}(a - f(a))$). These observations give us

Corollary 5.2. (i) *If $2A = 0$, then $\mathbb{Z}_2$-gradings correspond to idempotent derivations* (see [10]).

(ii) *If A is 2-torsion-free, then $\mathbb{Z}_2$-gradings correspond to automorphisms f such that $f^2 = \mathrm{id}$ and $a - f(a) \in 2A$ for all $a \in A$* (see [2]).

An examination of the proof of Theorem 5.1 shows that we have established a bit more than is stated: each grading of A defines a pair f, δ satisfying (ii) and conversely; likewise each grading defines a pair f, δ satisfying (iii) and conversely. In each case we have two inverse correspondences.

For instance, if we have a $\mathbb{Z}_2$-grading $A = A_0 + A_1$, the resulting f and δ ($f(a_0 + a_1) = a_0 - a_1; \delta(a_0 + a_1) = a_1$) (which satisfy (ii)) in turn define a grading $A = B_0 + B_1$, where $B_0 = \{a \in A : \delta(a) = 0\} = A_0$ (as $\delta(a)$ is the 1-component of a with respect to the original grading). Similarly $B_1 = A_1$.

If f is an automorphism with $f^2 = f$ and $\delta(a) = a - f(a)$ for all a as in the theorem,then by [5, Example 2] and the properties of δ established in the proof of the theorem,we have a D-structure whose maps are multiples of compositions of f and δ (and id), with some simplifications because f and δ commute.

On the other hand, if we start with f and δ satisfying (ii), we get a grading $A = A_0 + A_1$, where $A_0 = \{a \in A : \delta(a) = 0\}$ and $A_1 = \{a \in A : \delta(a) = a\}$. For this grading, we define F and Δ by setting $F(a_0 + a_1) = a_0 - a_1$ and $\Delta(a_0 + a_1) = a_1$ for all $a_0 \in A_0$, $a_1 \in A_1$. Then for all $a_0 \in A_0$, $a_1 \in A_1$, we have

$$\delta(a_0 + a_1) = \delta(a_0) + \delta(a_1) = 0 + a_1 = \Delta(a_0 + a_1) \quad \text{so} \quad \Delta = \delta.$$

But then, for each $a = a_0 + a_1 \in A$ we have

$$F(a) = F(a_0 + a_1) = a_0 - a_1 = a - 2a_1 = a - 2\Delta(a) = a - 2\delta(a) = f(a).$$

Thus $F = f$ and $\Delta = \delta$.

When A is 2-torsion-free, we get similar correspondences involving (i) and (iv). All of this is summarized in

Corollary 5.3 (to proof). *The constructions of the theorem yield inverse bijections between the set of* $\mathbb{Z}_2$*-gradings of a ring* A *with identity and pairs* f, δ *where* f *is an automorphism of* A *with* $f^2 = \mathrm{id}$ *and* δ *is an idempotent* (f, id)*-derivation in each of the following cases*:

(1) $f\delta = \delta f = -\delta$;

(2) $a = f(a) + 2\delta(a)$ *for all* $a \in A$.

When A *is* 2*-torsion free, there are such bijections when*

(3) $a - f(a) \in 2A$ *for all* $a \in A$.

The argument used to prove (iv) $\Rightarrow$ (iii) for 2-torsion-free rings in the theorem cannot be extended to rings in general, as the following example shows.

Example 5.4. Let A be the ring whose additive group is $\mathbb{Z}(2^\infty) \oplus \mathbb{Z}(2^\infty) \oplus \mathbb{Z}$ and whose multiplication is defined by the formula

$$(a, b, n)(c, d, m) = (ma + nc, mb + nd, nm).$$

It is easy to verify that this *is* a ring: in fact it is the standard unital extension of the zeroring on the direct sum of two copies of $\mathbb{Z}(2^\infty)$. Define $f\colon A \to A$ by setting $f((a, b, n)) = (b, a, n)$ for all a, b, n. Then f is an automorphism with $f^2 = f$. Also $(a, b, n) - f((a, b, n)) = (a - b, b - a, 0) \in 2A$ for all a, b, n, as $\mathbb{Z}(2^\infty)$ is a divisible group. Thus A satisfies (iv) of the theorem. Suppose there is an *additive endomorphism* g of A such that $(a, b, n) = f((a, b, n)) + 2g((a, b, n))$ for all a, b, n. If (a, b, n) has order 1 or 2, then $2g((a, b, n)) = 0$, so (a, b, n) is a fixed point of f. But $\mathbb{Z}(2^\infty)$ has an element r of order 2, so $(r, 0, 0)$ has order 2 in A, while $f((r, 0, 0)) = (0, r, 0) \neq (r, 0, 0)$. Hence there is no such g, so in particular there is no (f, id)-*derivation* to play this role.

This example also shows that the third pair of bijections in Corollary 5.3 do not exist for rings in general. The ring A of Example 5.4 *is* $\mathbb{Z}_2$-graded: $A = A_0 + A_1$, where $A_0 = \{(0, 0, n) : n \in Z\} \cong Z$ and $A_1 = \{(a, b, 0) : a, b \in \mathbb{Z}(2^\infty)\}$; the latter is isomorphic to the zeroring on the direct sum of two copies of $\mathbb{Z}(2^\infty)$. We have $A_1A_1 = \{0\} \subseteq A_0$, and the other requirements are clearly met.

We do not know of a non-$\mathbb{Z}_2$-gradable ring with an automorphism satisfying (iv) of Theorem 5.1.

We end with a description of the monoid ring (skew polynomial ring) associated with the "grading D-structure" referred to above (cf. [5, Example 2]). We retain our earlier notation.

Example 5.5. Let $A = A_0 + A_1$, $f(a_0 + a_1) = a_0 - a_1$, $\delta(a_0 + a_1) = a_1$ as in Theorem 5.1. Then for $m \geq n$ we have $\sigma_{mn} = \sigma_{x^m, x^n} = \binom{m}{n} f^n \delta^{m-n}$.

But $f^2 = \mathrm{id}$ and $\delta^2 = \delta$ so $\sigma_{mm} = \sigma_{x_m,x_m} = f$ if m is odd and id if m is even, while for $m > n$, $\sigma_{mn} = (-1)^n\delta$. In particular $\sigma_{00} = \mathrm{id}$, $\sigma_{10} = \delta$ and $\sigma_{11} = f$. In the resulting $A[x, \sigma]$ we have $xa = f(a)x + \delta(a) = (a_0 - a_1)x + a_1$ for all $a = a_0 + a_1 \in A$. For instance, in the case of $\mathbb{C}$ with the standard grading $\mathbb{C} = \mathbb{R} + \mathbb{R}i$, we have $x(r + si) = (r - si)x + si$ for all $r, s \in \mathbb{R}$.

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