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SUBNORMALIZING EXTENSIONS AND D-STRUCTURES

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Abstract If S is a subnormalizing extension [7] of a ring R with identity generated by a set which is a multiplicative submonoid of S and generates S as a free unital left R-module, then using the left and right module structures of S one can define a set of mappings $R \to R$ which satisfy all but (possibly) one of the requirements for a D-structure [2], [3], [4]. It remains unknown whether this remaining condition must in fact be satisfied, but we show that it is satisfied for a number of particular monoids and is at least partially satisfied in general. In the other direction it is known that many but by no means all D-structures are linked to subnormalizing extensions in this way.

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1. INTRODUCTION

A ring S with identity is a *subnormalizing extension* [7] of a ring R with the same identity 1 if S has a finite subset $\{x_1, x_2, \dots, x_n\}$ or a countably infinite subset $\{x_1, x_2, \dots\}$ such that

$$x_1 = 1,$$

$$S = \sum_i Rx_i = \sum_i x_i R$$

and for every i

$$\sum_{j=1}^{i} x_j R = \sum_{j=1}^{i} Rx_j.$$

Subnormalizing extensions are also called *extensions triangulaires* [6].

A D-*structure* [2], [3], [4] defined by a ring A with identity 1 and a monoid G with an identity e is a family of self-maps $\sigma_{x,y}$ of A, labelled by elements of G and satisfying the following conditions for all $a, b \in A$ and $x, y, z \in G$.

Condition (A)

(0) For each $x \in G$ and $a \in R$, we have $\sigma_{x,y}(a) = 0$ for almost all $y \in G$.

(i) Each $\sigma_{x,y}$ is an additive endomorphism.

(ii) $\sigma_{x,y}(ab) = \sum_{z\in G} \sigma_{x,z}(a)\sigma_{z,y}(b)$.

(iii) $\sigma_{xy,z} = \sum_{uv=z} \sigma_{x,u} \circ \sigma_{y,v}$.

(iv_1) $\sigma_{x,y}(1) = 0$ if $x \neq y$; (iv_2) $\sigma_{x,x}(1) = 1$;

(iv_3) $\sigma_{e,x}(a) = 0$ if $x \neq e$; (iv_4) $\sigma_{e,e}(a) = a$.

With this notation we shall sometimes refer briefly to the system as "the D-structure σ".

A D-structure defines a generalized monoid ring $A < G; \sigma >$ in which the multiplication is given by the formula

$$ax \cdot by = a \sum_{z\in G} \sigma_{x,z}(b)zy.$$

For example, skew polynomial rings arise this way. Many other examples can be found in [3], [4] and [5].

We shall generally retain all the above notation and use it without comment.

In some important cases $A < G; \sigma >$ is a subnormalizing extension of A [5]. On the other hand, if S is a subnormalizing extension of R such that the x_i form a multiplicative submonoid of S and generate S as a free unital left R-module we might ask how close S comes to being the generalized monoid ring defined by a D-structure. (The two stated conditions on the x_i are clearly necessary.) It turns out that S comes quite close: we can use S to define a set of self-maps of R which satisfy all the requirements for a D-structure except (possibly) (iii). For some monoids (i.e. the set of x_i as a submonoid of S) (iii) must also be satisfied. Moreover, in such cases, the generalized monoid ring defined by the resulting D-structure is the original subnormalizing extension, ([5], Theorem 4.1). It is not known whether this is true of all monoids. In this note we prove a general theorem to the effect that a weaker form of (iii) is universally satisfied and provide further examples of (classes of) monoids for which (iii) *is* satisfied. First, though, we have to describe the maps associated with a subnormalizing extension. From now on we shall only consider subnormalizing extensions for which the two stated conditions on the x_i hold, i.e.

the x_i form a multiplicative submonoid of S and S is a free left unital R-module freely generated by the x_i.

For each x_i and each $r \in R$ there are uniquely determined $r_{i1}, r_{i2}, \dots, r_{ii} \in R$ such that

$$x_i r = r_{i1}x_1 + r_{i2}x_2 + \cdots + r_{ii}x_i.$$

(They are uniquely determined because S is free.) If we re-name r_{ij} as $\sigma_{x_i,x_j}(r)$, this defines maps $\sigma_{x_i,x_j} : R \to R$ for all i, j with $i \geq j$. For $i < j$ we let σ_{x_i,x_j} be the zero map. These maps satisfy all conditions for a D-structure except possibly (iii) ([5], Section 4) and if the x_i form a cyclic group of order 2 or a two element (semi)lattice, then (iii) is satisfied ([5], Examples 4.2 and 4.3).

2. RESULTS AND EXAMPLES

We shall say that a monoid G *has automatic D-structures* if whenever $G = \{x_1, x_2, \ldots, x_n\}$ or $\{x_1, x_2, \ldots\}$ for a subnormalizing extension S of a ring R, the associated maps σ_{x_i,x_j} form a D-structure σ (and hence $S = R < G; \sigma >$). The following theorem reduces the number of calculations needed to show that a monoid has automatic D-structures.

Theorem 2.1. *Let S be a subnormalizing extension of a ring R with the same identity* 1, $S = \sum_{i=1}^{n} Rx_i$ *or* $S = \sum_{i\in\mathbb{Z}^+} Rx_i$, *where* $x_1 = 1$. *Suppose further that*
(i) $\{x_1, x_2, \cdots, x_n\}$ *or* $\{x_1, x_2, \cdots\}$ *is a multiplicative submonoid of S and*
(ii) S is a free left R-module on the x_i.

For $r \in R$ and each i let $x_i r = \sum_{p=1}^{i} \sigma_{x_i,x_p}(r)x_p$ *and for $p > i$ let σ_{x_i,x_p} be the zero map.*

If $x_i x_j = x_m$, then

$$\sigma_{x_i x_j,x_\ell} = \sigma_{x_m,x_\ell} = \sum_{x_s x_t = x_\ell} \sigma_{x_i,x_s} \circ \sigma_{x_j,x_t}$$

for $\ell \leq m$.

Proof. Let $x_i x_j = x_m$. Then for all $r \in R$ we have

$$\sum_{\ell\leq m} \sigma_{x_m,x_\ell}(r)x_\ell = x_m r = x_i x_j \cdot r = x_i \cdot x_j r = x_i \sum_{t\leq j} \sigma_{x_j,x_t}(r)x_t$$

$$= \sum_{t\leq j}(x_i \sigma_{x_j,x_t}(r))x_t = \sum_{t\leq j}(\sum_{s\leq i} \sigma_{x_i,x_s}(\sigma_{x_j,x_t}(r)x_s)x_t.$$

Now as for $t > j$ or $s > i$ we have $\sigma_{x_i,x_s} \circ \sigma_{x_j,x_t}(r) = 0$, we therefore have, for each $\ell \leq m$, equating coefficients,

$$\sigma_{x_m,x_\ell}(r)x_\ell = \sum_{x_s x_t = x_\ell} \sigma_{x_i,x_s} \circ \sigma_{x_j,x_t}(r).$$

Since r is arbitrary we therefore have

$$\sigma_{x_i x_j,x_\ell} = \sigma_{x_m,x_\ell} = \sum_{x_s x_t = x_\ell} \sigma_{x_i,x_s} \circ \sigma_{x_j,x_t}$$

for all $\ell \leq m$. ∎

Example 2.1. *The free monoid F on a single generator x has automatic D-structures. Since $F = \{e, x, x^2, \dots\}$ we standardize notation to fit our discussion by writing $x^n = x_{n+1}$ for all n (including $e = x^0$). If $x_i x_j = x_m$ and $m < \ell$, then $x^{m-1} = x_m = x_i x_j = x^{i-1}x^{j-1} = x^{i+j-2}$, so $i + j - 2 = m - 1 < \ell - 1$. Hence if $x_r x_s = x_\ell$ then as above $r + s - 2 = \ell - 1 > i + j - 2$ whence $i + j < r + s$. But then $i < r$ or $j < s$, so $\sigma_{x_i,x_r} \circ \sigma_{x_j,x_s} = 0$ and thus*

$$\sum_{x_r x_s = x_\ell} \sigma_{x_i,x_r} \circ \sigma_{x_j,x_s} = 0 = \sigma_{x_m,x_\ell} = \sigma_{x_i x_j,x_\ell},$$

and by the theorem this is all we need.

A subnormalizing extension using F will be some kind of generalized polynomial ring. Skew polynomial rings ([1], pp. 34-40) are subnormalizing extensions. See also Propositions 6.1-6.5 of [3].

Example 2.2. *The countably infinite chain $C = \{x_1, x_2, \dots\}$ with the order type of the natural numbers, when viewed as a semilattice with operation $x_i x_j = x_{\max\{i,j\}}$ is a monoid which has automatic D-structures. Clearly x_1 is an identity. If $x_i x_j = x_m$ and $m < \ell$, then $m = \max\{i, j\}$ and if $x_r x_s = x_\ell$, we have $\ell = \max\{r, s\}$ so r or s is equal to ℓ. Hence either $r > m \geq i$ or $s > m \geq j$, so $\sigma_{x_i,x_r} \circ \sigma_{x_j,x_s} = 0$. Thus*

$$\sum_{x_r x_s = x_\ell} \sigma_{x_i,x_r} \circ \sigma_{x_j,x_s} = 0.$$

Again, an appeal to the theorem completes the proof.

A *right zero semigroup* is a semigroup in which $xy = y$ for all x, y. Clearly (except for the one-element semigroup) a right zero semigroup cannot be a monoid. We get a monoid by adjoining an identity, and such monoids provide our next example.

Example 2.3. *Let $H = \{x_1, x_2, \dots\}$ be a countably infinite monoid where x_1 is the identity and $\{x_i : i > 1\}$ is a right zero semigroup. Then H has automatic D-structures. We have to consider maps $\sigma_{x_i x_j,x_k}$ where $x_i x_j = x_p$ and $p < k$. Note that then $k \geq 2$. There are several cases to consider, but in every case, $x_r x_s = x_k$ precisely when $s = k$ or $r = k$ and $s = 1$; in particular r or s is equal to k.*

For $i = j = 1$, $\sigma_{x_1,x_r} \circ \sigma_{x_1,x_s} = 0$ in all cases.

For $i = 1$ and $j \neq 1$, $\sigma_{x_1,x_r} \circ \sigma_{x_j,x_k} = 0 = \sigma_{x_1,x_k} \circ \sigma_{x_j,x_1}$ (as $j\ (= p) < k$).

For $i \neq 1$ and $j = 1$, $\sigma_{x_i,x_r} \circ \sigma_{x_1,x_k} = 0 = \sigma_{x_i,x_k} \circ \sigma_{x_1,x_1}$ (as $i < k$).

Now consider $i, j, k > 1$. Since $x_i x_j = x_j$ in this case, we have $j < k$.

If $k > i$ *then* $\sigma_{x_i,x_r} \circ \sigma_{x_j,x_k} = 0 = \sigma_{x_i,x_k} \circ \sigma_{x_j,x_1}$.

There remains the case $1 < j < k \leq i$. *We need to make a digression to take care of this case.*

For each element a *of the ring of which* H *generates a subnormalizing extension we have*

$$x_j a = \sum_{t=1}^{j} \sigma_{x_j,x_t}(a) x_t.$$

Also

$$x_j a = x_i x_j \cdot a = x_i \cdot x_j a = x_i \sum_{t=1}^{j} \sigma_{x_j,x_t}(a) x_t = \sum_{t=1}^{j} (x_i \sigma_{x_j,x_t}(a)) x_t$$

$$= (\sum_{v=1}^{i} \sigma_{x_i,x_v} \circ \sigma_{x_j,x_t}(a)) x_1 + (\text{multiples of} \quad x_2) + \cdots + (\text{multiples} \quad \text{of} \quad x_j).$$

Equating coefficients of x_k *in these two expressions (remembering that* $j < k \leq i$*) we get*

$$\sigma_{x_i,x_k} \circ \sigma_{x_j,x_1} = 0.$$

But also $\sigma_{x_i,x_r} \circ \sigma_{x_j,x_k} = 0$ *for every value of* r.

In all cases we have shown that if $x_i x_j = x_p$ *and* $p < k$ *then*

$$\sum_{x_r x_s = x_k} \sigma_{x_i,x_r} \circ \sigma_{x_j,x_s} = 0 = \sigma_{x_i x_j, x_k}.$$

As usual the remaining cases are covered by the theorem.

Example 2.4. *The cyclic group of order* 3 *has automatic D-structures. We omit the details; even with the simplification provided by the theorem a lot of calculation is needed, similar to that used in the very last case in Example 2.3. Cf. also the corresponding calculations for the cyclic group of order* 2 *in [5] (Example 4.2).*

The arguments used in Examples 2.2 and 2.3 work verbatim to show that finite chains and finite right zero semigroups with identity adjoined have automatic D-structures.

As pointed out earlier, it remains an open question whether all monoids have automatic D-structures. The conditions imposed on subnormalizing extensions, that they are free modules and that the generators form a submonoid, are strong. At the same time, a generalized monoid ring defined by a D-structure can be quite different from the standard monoid ring: for instance, over a field K of characteristic 0, the (first) Weyl algebra, which is simple, is a generalized monoid ring over the polynomial ring $K[X]$.

Recognizing a potential D-structure in a subnormalizing extension *may* make it easier to show that there really is a D-structure there by obviating the need to check (*iii*) of the defining conditions to establish associativity of the generalized monoid ring. However checking associativity of a potential subnormalizing extension might be just as hard.

Given a ring A with identity, a monoid G and self-maps $\sigma_{x,y}$ of A $(x, y \in G)$ which satisfy all the requirements for a D-structure except possibly (*iii*), we can still use the maps as before to define an extension of A which we shall still call $A < G; \sigma >$, though it may not be associative. Although if (*iii*) is true then the extension *is* associative, and (*iii*) was used in [3] just to prove associativity, it is not known whether (*iii*) is *necessary* for associativity. As (*iii*) can be difficult (or at least time-consuming) to verify, a weaker condition which yet assures associativity would be useful. The following example has connections with several aspects of this discussion.

Example 2.5. *Let A be a ring with identity, δ a derivation on A. For a cyclic group $G = \{e, x\}$ of order 2, let*

$$\sigma_{e,e} = \sigma_{x,x} = id; \quad \sigma_{e,x} = 0; \quad \sigma_{x,e} = \delta.$$

If $2\delta = 0 = \delta^2$ we get a D-structure ([5], Example 3.1). Without imposing any conditions on the derivation δ, suppose $A < G; \sigma >$ is associative. Then for all $c \in A$, we have

$$x(x \cdot cx) = x(1\sigma_{x,x}(c)xx + 1\sigma_{x,e}(c)ex) = x(ce + \delta(c)x) = x \cdot ce + x \cdot \delta(c)x$$

$$= 1\sigma_{x,x}(c)xe + 1\sigma_{x,e}(c)ee + 1\sigma_{x,x}(\delta(c))xx + 1\sigma_{x,e}(\delta(c))ex$$

$$= cx + \delta(c)e + \delta(c)e + \delta^2(c)x = (c + \delta^2(c))x + 2\delta(c)e$$

and

$$x^2 \cdot cx = e \cdot cx = 1\sigma_{e,e}(c)ex + 1\sigma_{e,x}(c)xx = cx.$$

Comparing the two expressions we see that (as c is arbitrary) $2\delta = 0 = \delta^2$. Now in Example 3.1 of [5] it was shown that these conditions on δ imply that ((iii) is satisfied and hence) $A < G; \sigma >$ is associative. Thus in this very restricted setting, where pretty much everything is defined by a derivation δ, the following are equivalent:

1 $A < G; \sigma >$ is associative;
2 $2\delta = 0 = \delta^2$;
3 (iii) is satisfied.

Also, as $xa = 1x \cdot ae = 1\sigma_{x,x}(a)xe + 1\sigma_{x,e}(a)ee = ax + \delta(a)e$, we see that $A < G; \sigma >$ is a "non-associative subnormalizing extension" of A.

How non-associative can the ring be? We have

$$(ax)^2 ax = a\delta(a)ae + (a^3 + a(\delta(a))^2)x$$

and

$$ax(ax)^2 = (a\delta(a)a + 2a^2\delta(a))e + (a^3 + a(\delta(a))^2 + a^2\delta^2(a))x.$$

Thus if there is some $a \in A$ such that $2a^2\delta(a) \neq 0$ or $a^2\delta^2(a) \neq 0$ then the ring is not even third-power-associative. For such a case, take A to be the polynomial ring $\mathbb{R}[X]$, let δ be formal differentiation and $a = 2X$.

One aspect of our discussion of automatic D-structures needs some comment. The generators of a subnormalizing extension are listed in a certain way and the defining properties of a subnormalizing extension are expressed in terms of the generators *and* the listing. (We use the word "listing" rather than "order" to avoid getting mixed up with the order of ordered monoids.) It is conceivable that a ring extension which is not subnormalizing for a given listing of the generators might become so if the generators are listed in a different way.

Each of the monoids we have examined has its elements listed in just one way (matched in associated subnormalizing extensions). Could a monoid cease to have automatic D-structures if its elements were listed in a different way? Possibly; we don't know.

The elements of a monoid of order 2 can only be listed in one way as the identity must come first. There are no problems here. For a cyclic group G of order 3, $G = \{e, x,^2\}$, we have always used the listing $x_1 = e$, $x_2 = x$, $x_3 = x^2$. The other possible listing is e, x^2, x. But as x^2 also generates G and $(x^2)^2 = x$, this is not essentially different. If $H = \{x_1, x_2, \ldots x_n, \ldots\}$ where $\{x_n : n > 1\}$ is a right zero semigroup, then for every permutation ρ of $\mathbb{Z}^+$ which fixes 1, we get an automorphism

$$H \to H; \quad (\forall n) \;\; x_n \mapsto x_{\rho(n)}.$$

The cases of the infinite chain with identity as lowest element and the free monoid on one generator are less clear. However, these are both *ordered monoids* and the listings we have used correspond to their orders. Thus we can say that *as ordered monoids* they have automatic D-structures. Finite chains with identity as smallest element are also ordered monoids.

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