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FURTHER REMARKS ON ELEMENTARY RADICALS AND ASSOCIATED FILTERS OF IDEALS

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In memory of Yuri Ryabukhin

Abstract

Ryabukhin showed that there is a correspondence between elementary radical classes of rings and certain filters of ideals of the free ring on one generator, analogous to the Gabriel correspondence between torsion classes of left unital modules and certain filters of left ideals of the coefficient ring. This correspondence is further explored here. All possibilities for the intersection of the ideals in a filter are catalogued, and the connections between filters and other ways of describing elementary radical classes are investigated. Some generalisations to nonassociative rings and groups are also presented.

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1. Introduction

Let F be the free associative ring on a single generator x, that is, $F = x\mathbb{Z}[x]$. Elements of F are denoted by symbols such as f or $f(x)$ as convenient, and the composition of f and g is written as $f \circ g$ or $f(g)$. For $f \in F$ and an element a of a ring A, $f(a)$ denotes the element of A obtained by the substitution of a for x in f. This gives an action of F on A, which shares some features with the action of a ring with identity on its modules. For an element a of a ring A, we denote by $[a]$ the subring generated by a. We let

$$(0 * a) = \{f \in F : f(a) = 0\},$$

and, more generally,

$$(I * a) = \{f \in F : f(a) \in I\}$$

for every $I \triangleleft A$. In our analogy between F acting on rings and a ring with identity acting on modules, $(0 * a)$ plays the part of the annihilator of a module. For every element a of every ring A there is a homomorphism

$$F \to A; \quad x \mapsto a,$$

whose image is $[a]$ and whose kernel is $(0 * a)$. This gives the following statement.

PROPOSITION 1.1. *For every element a of a ring A, $(0 * a) \triangleleft F$.*

The following concept was introduced by Ryabukhin [11]. A set Φ of ideals of F is called a *radical filter* if it satisfies the following conditions.

(1) If $G \in \Phi$ and $G \subseteq T \triangleleft F$, then $T \in \Phi$.
(2) If $G \in \Phi$ and $f \in F$, then $(G * f) \in \Phi$.
(3) If $G \in \Phi, H \triangleleft F$ and $(H * g) \in \Phi$ for every $g \in G$, then $H \in \Phi$.

If $H \triangleleft F$ and $G \in \Phi$, then $H \subseteq G + H \in \Phi$ by (1). If $f(x) = a_1x + a_2x^2 + \cdots + a_nx^n \in F$, then for $g \in G$ and $h \in H$,

$$\begin{aligned} f(g+h) &= a_1(g+h) + a_2(g^2 + 2gh + h^2) + \cdots + a_n(g^n + ng^{n-1}h + \cdots + ngh^{n-1} + h^n) \\ &= f(g) + \text{ an element of } H, \end{aligned}$$

whence $f(g) \in H$ if and only if $f(g+h) \in H$. Thus, in (3) it may be assumed that $H \subseteq G$. If $G, H \in \Phi$ then, using the isomorphism $G/(G \cap H) \cong (G+H)/H$, we see that for $g \in G$,

$$(G \cap H * g) = (0 * (g + G \cap H)) = (0 * (g + H)) = (H * g) \in \Phi,$$

whence, by (3), if $G, H \in \Phi$ then $G \cap H \in \Phi$. (Thus, Φ really is a filter.) A radical class $\mathcal{R}$ (in the Kurosh–Amitsur sense) is said to be *elementary* if it satisfies the condition

$$A \in \mathcal{R} \iff \forall a \in A, [a] \in \mathcal{R}.$$

Note that an elementary radical class is *strongly hereditary*: subrings of rings in $\mathcal{R}$ are themselves in $\mathcal{R}$. Elementary radical classes have several different names in the literature; Ryabukhin [11] called them *semistrictly hereditary (polustrogo nasledstvennyi)*. Our principal interest is the connection between elementary radical classes and radical filters.

THEOREM 1.2 (Ryabukhin [11]). *For an elementary radical class $\mathcal{R}$ and a radical filter Φ, let $\Phi_{\mathcal{R}} = \{(0 * a) : a \in A \in \mathcal{R}\}$ and $\mathcal{R}_\Phi = \{A : a \in A \Rightarrow (0 * a) \in \Phi\}$. Then*

(i) *$\Phi_{\mathcal{R}}$ is a radical filter;*
(ii) *$\mathcal{R}_\Phi$ is an elementary radical class; and*
(iii) *the correspondences $\mathcal{R} \mapsto \Phi_{\mathcal{R}}$ and $\Phi \mapsto \mathcal{R}_\phi$ define inverse bijections between the set of elementary radical classes and the set of radical filters.*

The following result is useful for proving Theorem 1.2 and elsewhere.

PROPOSITION 1.3. *Let $\mathcal{R}$ be an elementary radical class and define $\Phi_{\mathcal{R}}$ as in Theorem 1.2. Then $\Phi_{\mathcal{R}} = \{G \triangleleft F : F/G \in \mathcal{R}\}$.*

PROOF. If $G \in \Phi_{\mathcal{R}}$ then $G = (0 * b)$ for some $b \in B \in \mathcal{R}$. But $[b] \in \mathcal{R}$, so that we have $F/G = F/(0 * b) \cong [b] \in \mathcal{R}$. Conversely, if $G \triangleleft F$ and $F/G \in \mathcal{R}$, let $\overline{x} = x + G$. Then

$$G = \{f \in F : f(x) \in G\} = (0 * \overline{x}),$$

where $\overline{x} \in F/G \in \mathcal{R}$. □

REMARK 1.4. As the radical filters form a set, so do the elementary radical classes.

Radical filters of ideals of F and their relationship to elementary radical classes closely resemble the *idempotent topologising filters* of left ideals of a ring R with identity introduced by Gabriel [1] a few years earlier and their connections with hereditary radical classes (torsion classes) of (left unital) R-modules. Since modules form an abelian category, hereditary radical classes of modules are elementary in the obvious sense. Although Gabriel's definition of his filters was a bit different, there is an equivalent formulation very like that of radical filters (see, for example, the book by Mishina and Skornyakov [8, Sections 0 and 2]). In more recent literature, idempotent topologising filters are commonly called radical filters.

2. Results

If an elementary radical class $\mathcal{R}$ is closed under direct products, it must be a variety and hence a *semi-simple-radical class*, that is, it is the semi-simple class corresponding to another radical class (see [6, Section 3.20]). In this case, there is an ideal K of F which is a T-ideal (invariant under endomorphisms of F) such that $\mathcal{R}$ is the class of rings satisfying the identities $\{f \approx 0 : f \in K\}$.

THEOREM 2.1. *Let $\mathcal{R}$ be an elementary radical class. The following conditions are equivalent.*

(i) *$\mathcal{R}$ is closed under direct products.*
(ii) *$\mathcal{R}$ is a variety.*
(iii) *$\mathcal{R}$ is a semi-simple-radical class.*
(iv) $\bigcap\{G : G \in \Phi_{\mathcal{R}}\} \in \Phi_{\mathcal{R}}$.
(v) *$K = \bigcap\{G : G \in \Phi_{\mathcal{R}}\}$ is a T-ideal and K is the ideal of F generated by the set $\{\ell \circ k : \ell, k \in K\}$.*

PROOF. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) as noted.
(iii) $\Rightarrow$ (iv). By (iii), $\mathcal{R}$ is the variety generated by a finite set of finite fields. Let $A = K_1 \oplus K_2 \oplus \cdots \oplus K_n$ be the direct sum of all fields in $\mathcal{R}$, and let $[a_i] = K_i$ for each i. Then

$$(0 * (a_1, a_2, \ldots, a_n)) \in \Phi_{\mathcal{R}}$$

and, as each ring in $\mathcal{R}$ is a subdirect product of (copies of) the fields K_i,

$$(0 * (a_1, a_2, \ldots, a_n)) \subseteq (0 * b) \quad \text{for all } b \in B \in \mathcal{R}.$$

This gives

$$(0 * (a_1, a_2, \ldots, a_n)) = \bigcap\{G \triangleleft F : G \in \Phi_{\mathcal{R}}\} \in \Phi_{\mathcal{R}}.$$

(iv) $\Rightarrow$ (v). Let $K = \bigcap\{G \triangleleft F : G \in \Phi_{\mathcal{R}}\}$. Then $F/K \in \mathcal{R}$, so for all $f \in F$,

$$(K * f) = (0 * (f + K)) \in \Phi_{\mathcal{R}}.$$

Thus, $K \subseteq (K * f)$, which means that $k \circ f = k(f) \in K$ for all $k \in K, f \in F$.

Let α be an endomorphism of F and let $\alpha(x) = c_1x + c_2x^2 + \cdots + c_nx^n = h(x) \in F$. Then, for any $k = b_1x + b_2x^2 + \cdots + b_mx^m \in K$,

$$\begin{aligned}\alpha(k) = \alpha(b_1x + b_2x^2 + \cdots + b_mx^m) &= b_1\alpha(x) + b_2\alpha(x)^2 + \cdots + b_m\alpha(x)^m \\ &= k(\alpha(x)) = k(h) \in K.\end{aligned}$$

Thus, K is preserved by endomorphisms of F and is a T-ideal. Since $\mathcal{R}$ is defined as a variety by identities in one variable (for an explicit description, see the theorem in [5]), K is the T-ideal which defines R.

Now let $K \circ K$ be the ideal of F generated by all $k_1 \circ k_2$ for $k_1, k_2 \in K$. In $K/K \circ K$, for each $k \in K$, $\ell \circ k \in K \circ K$ for all $\ell \in K$, so that

$$(0 * (k + K \circ K)) = (K \circ K * k) \supseteq K \in \Phi_{\mathcal{R}}$$

and $K/K \circ K \in \mathcal{R}$. But also $(F/K \circ K)/(K/K \circ K) \cong F/K \in \mathcal{R}$, so we conclude that $F/K \circ K \in \mathcal{R}$, whence $K \subseteq K \circ K \subseteq K$.

(v) $\Rightarrow$ (i). Let Φ be the set of ideals of F which contain K. We will show that Φ is a radical filter. Condition (1) is obvious. If $K \subseteq G \triangleleft F$ and $f = a_1x + a_2x^2 + \cdots + a_nx^n$ then there is an endomorphism α of F such that $\alpha(x) = f$. Hence, for every $k \in K$,

$$k(f) = k(\alpha(x)) = \alpha(k) \in K \subseteq G,$$

so $K \subseteq (G * f)$ and this is (2) for Φ. Finally, if $K \subseteq G \triangleleft F$, $H \triangleleft F$ and $(H * g) \supseteq K$ for all $g \in G$, then $k \circ g \in H$ for all $k \in K$ and $g \in G$, so in particular $k \circ \ell \in H$ for all $k, \ell \in K$. But then $K \subseteq H$, and so Φ satisfies (3) and is therefore a radical filter. Now $K = \bigcap\{G \triangleleft F : G \in \Phi_{\mathcal{R}}\}$, so $\Phi_{\mathcal{R}} \subseteq \Phi$. But by (1) in the definition of a radical filter, every radical filter which contains K must contain Φ, so in particular $\Phi \subseteq \Phi_{\mathcal{R}}$ and the two filters are equal.

If $a_\lambda \in A_\lambda \in \mathcal{R}$ for all λ in some index set Λ, then in $\prod_{\lambda\in\Lambda} A_\lambda$,

$$(0 * (a_\lambda)_\Lambda) = \{f \in F : f((a_\lambda)_\Lambda) = 0\} = \{f \in F : (f(a_\lambda))_\Lambda = 0\} = \bigcap_{\lambda\in\Lambda}(0 * a_\lambda) \supseteq K,$$

so $(0 * (a_\lambda)_\Lambda) \in \Phi_{\mathcal{R}}$. Hence, $\prod_{\lambda\in\Lambda} A_\lambda \in \mathcal{R}$. □

COROLLARY 2.2. *The following conditions are equivalent for an ideal K of F.*

(i) *$\{G \triangleleft F : K \subseteq G\}$ is a radical filter.*
(ii) *K is a T-ideal and $K \circ K = K$.*

This result is a corollary of the proof of Theorem 2.1. Note that since the correspondence between radical filters and elementary radical classes is bijective, we also have a bijection between semi-simple radical classes and T-ideals K of F for which $K = K \circ K$.

Analogously, it was shown by Jans [7] that a hereditary radical class of modules is closed under direct products if and only if there is an idempotent ideal I such that the corresponding filter of left ideals is $\{L : I \subseteq L\}$.

We shall examine the properties of the intersection of all ideals in a radical filter a bit further. Let $\mathcal{R}$ be any elementary radical class and let $J = \bigcap\{G \triangleleft F : G \in \Phi_{\mathcal{R}}\}$.

PROPOSITION 2.3. *If $\mathcal{R}$ contains some nonzero nilpotent rings then $J = 0$.*

PROOF. Since $\mathcal{R}$ is strongly hereditary, if it contains nonzero nilpotent rings it contains a nonzero torsion-free nilpotent ring or one which is a reduced p-ring for some prime p (and the former implies the latter anyway). But then by [2, Theorem 1.2] there is no nontrivial polynomial identity satisfied by every ring in $\mathcal{R}$. But $f(a) = 0$ for all $a \in A \in \mathcal{R}$ for all $f \in J$, so $J = 0$. □

A ring A is *periodic* if for each $a \in A$ there is an integer $n > 1$ such that $a^n = a$.

PROPOSITION 2.4. *If $J \neq 0$ then every ring in $\mathcal{R}$ is periodic.*

PROOF. Since $J \neq 0$, the previous result says that $\mathcal{R}$ contains no nonzero nilpotent rings. But if $a \in A \in \mathcal{R}$ then $\mathcal{R}$ contains $[a]$ and hence also $[a]/[a]^2$. Since the latter is nilpotent, $[a] = [a]^2$, that is, $a = r_2a^2 + r_3a^3 + \cdots$ for some integers r_i. Since this is true for all $a \in A$, by [12, Corollary 3.5] or [9, Theorem 13.2], A is periodic. □

In what follows, $GF(p^n)$ denotes the field with p^n elements.

Let $\mathcal{R}$ be an elementary radical class and let $\mathcal{F}$ be the set of fields in $\mathcal{R}$. Then $\mathcal{F}$ is strongly hereditary. Let $\mathcal{V}$ be the variety generated by $\mathcal{F}$. Since each ring in $\mathcal{R}$ is a subdirect product of periodic fields which must be in $\mathcal{F}$, we have $\mathcal{R} \subseteq \mathcal{V}$.

If $\mathcal{F}$ is finite, so that all fields in $\mathcal{F}$ are finite, then any ring $A \in \mathcal{V}$ is a subdirect product of fields in $\mathcal{F}$. By [12, Proposition 3.7], each finitely generated subring, so in particular each singly generated subring, is a finite direct sum of fields in $\mathcal{F}$ and hence in $\mathcal{R}$. Since $\mathcal{R}$ is elementary, A is in $\mathcal{R}$, so in this case $\mathcal{R} = \mathcal{V}$ and J is the T-ideal defining $\mathcal{V}$.

If $\mathcal{F}$ is infinite but contains only finite fields, then there are fields in $\mathcal{F}$ of infinitely many characteristics $p_1 < p_2 < \cdots < p_n < \cdots$, and as $\mathcal{F}$ is strongly hereditary it must contain $GF(p_n)$ for each n. Suppose J contains some $f = a_1x + a_sx^2 + \cdots + a_mx^m$. Then there exists a t such that $m < p_t$. Now $a_1y + a_2y^2 + \cdots + a_my^m = 0$ for every $y \in GF(p_t)$. Reducing all coefficients modulo p_t gives a polynomial $\overline{a}_1 + \overline{a}_2x^2 + \cdots + \overline{a}_mx^m \in GF(p_t)[x]$ with $p_t > m$ roots. Hence, the polynomial is zero, which means that the original coefficients $a_1, a_2, \ldots, a_m$ are all divisible by p_t. But in the same way the coefficients are also divisible by $p_{t+1}, p_{t+2}, \ldots$. Hence, the coefficients are all 0 and thus $J = 0$.

If $\mathcal{F}$ contains an infinite field P of characteristic p then, as in the previous argument, every $f \in J$ induces a polynomial over P with 'too many roots', whence $f \in pF$. It follows that if $\mathcal{F}$ contains infinite fields of infinitely many characteristics then $J = 0$. It is clear from what we have already said that if $\mathcal{F}$ contains infinite fields of finitely many characteristics and infinitely many finite fields of other characteristics, then $J = 0$ still holds.

There remains only the case where $\mathcal{F}$ contains infinite fields of finitely many characteristics $q_1, q_2, \ldots, q_n$ and at most finitely many finite fields $P_1, P_2, \ldots, P_m$ of characteristic $p_1, p_2, \ldots, p_m$, respectively, where no p_j is a q_i. If there are some P_j, let $\mathcal{W}$ be the variety generated by $\{P_1, P_2, \ldots P_m\}$ and let W be the corresponding T-ideal. Then $J = W \cap q_1 q_2 \cdots q_n F$. But for any $f \in \mathcal{F}$, if $q_1 q_2 \cdots q_n f \in W$ then, as $p_1 p_2 \cdots p_m f \in W$ and $q_1 q_2 \cdots q_n$ and $p_1 p_2 \cdots p_m$ are relatively prime, it follows that $f \in W$. Hence, $W \cap q_1 q_2 \cdots q_n F = q_1 q_2 \cdots q_n W$. Finally, if there are no P_j then $J = q_1 q_2 \cdots q_n F$.

In summary, we have proved the following result.

Theorem 2.5. *Let $\mathcal{R}$ be an elementary radical class and $\mathcal{R} \neq \{0\}$.*

(i) *If $\mathcal{R}$ contains nonzero nilpotent rings then $\bigcap \Phi_{\mathcal{R}} = 0$. Otherwise $\mathcal{R}$ consists of periodic rings and contains a set $\mathcal{F}$ of fields.*
(ii) *If $\mathcal{F}$ is finite then $\mathcal{R}$ is a semi-simple-radical class and $\bigcap \Phi_{\mathcal{R}}$ is the T-ideal corresponding to $\mathcal{R}$ as a variety.*
(iii) *If $\mathcal{F}$ contains infinite fields with finitely many characteristics $q_1, q_2, \ldots, q_n$ and finitely many finite fields $P_1, P_2, \ldots, P_m$ of characteristics other than $q_1, q_2, \ldots, q_n$, then $J = q_1 q_2 \cdots q_n W$.*
(iv) *If $\mathcal{F}$ contains infinite fields of finitely many characteristics $q_1, q_2, \ldots, q_n$ and no finite fields of other characteristics, then $J = q_1 q_2 \cdots q_n F$.*
(v) *In all other cases, $J = 0$.*

We do not need a whole radical filter to describe an elementary radical class. For example, if $\mathcal{R}$ is an elementary radical class and Θ is a subset of $\Phi_{\mathcal{R}}$ such that for each $G \in \Phi_{\mathcal{R}}$ there is an $H \in \Theta$ with $H \subseteq G$, then it is not difficult to show that

$$\mathcal{R} = \{A : (\forall a \in A)(\exists H \in \Theta)(H \subseteq (0 * a)\}.$$

However, there is another way of characterising an elementary radical class which uses even fewer ingredients. If S is a $\circ$-subsemigroup of F, then

$$\mathcal{R}_S = \{A : (\forall a \in A)(\exists f \in S)(f(a) = 0)\}$$

is an elementary radical class. For example, $\mathcal{R}_S$ is closed under extensions because S is closed under $\circ$.

Many examples are given in [4]. For instance, if $S = \{x^n : n = 1, 2, 3, \ldots\}$ then $\mathcal{R}_S = \mathcal{N}$ (the nil radical class).

What is the radical filter associated with a radical class $\mathcal{R}_S$? Which elementary radical classes can be represented by semigroups?

We know nothing about the second question. In particular, there is no elementary radical class which has been shown not to have a semigroup representation. We can say a bit more about the first. Since radical filters are closed under 'getting bigger', the following set of ideals seems promising. Let

$$\Phi_S = \{G \triangleleft F : G \cap S \neq \emptyset\}.$$

Sometimes this is the required radical filter. In what follows, we shall indicate the principal ideal of F generated by an element f by (f).

PROPOSITION 2.6. *Let $S = \{x^n : n = 1, 2, 3, \ldots\}$. Then $\Phi_S = \Phi_{\mathcal{N}}$.*

PROOF. We first prove that Φ_S is a radical filter. Condition (1) is obvious. If $x^n \in G \in \Phi_S$ and $f = a_1x + a_2x^2 + \cdots + a_mx^m \in F$ then

$$f^n = (a_1x + \cdots + a_mx^m)^n = a_1^n x^n + (\text{terms in higher powers of } x) \in (x^n) \subseteq G.$$

Hence $x^n \in (G * f)$, so $(G * f) \in \Phi_S$ and (2) is satisfied.

Again take $x^n \in G \in \Phi_S$ and let H be an ideal of F such that $H \subseteq G$ and $(H * g) \in \Phi_S$ for every $g \in G$. In particular, $(H * x^n) \in \Phi_S$. Let x^m be in $(H * x^n)$. Then $x^{mn} = (x^n)^m \in H$, so $H \in \Phi_S$. This proves (3), so Φ_S is a radical filter.

Hence, there is an elementary radical class $\mathcal{R}_{\Phi_S}$. If A is in this class and $a \in A$ then $(0 * a) \in \Phi_S$, so $x^n \in (0 * a)$ for some n, that is, $a^n = 0$ and it follows that A is nil. Conversely, if $b \in B \in \mathcal{N}$ then $b^m = 0$ for some m, which means that $x^m \in (0 * b)$ and so $(0 * b) \in \Phi_S$. We conclude that $\mathcal{N} = \mathcal{R}_{\Phi_S}$. But then, using the bijections between elementary radical classes and radical filters, $\Phi_{\mathcal{N}} = \Phi_{\mathcal{R}_{\Phi_S}} = \Phi_S$. □

Similarly, for $S = \{nx : n = 1, 2, 3, \ldots\}$, which defines the elementary radical class $\mathcal{T}$ of torsion rings, or $S = \{mx^n : m, n = 1, 2, 3, \ldots\}$, which defines the elementary radical class $\mathcal{T} \circ \mathcal{N}$ of rings A for which $A/\mathcal{T}(A) \in \mathcal{N}$, $\Phi_S = \Phi_{\mathcal{R}_S}$.

The next theorem follows by arguments like those in the proof of Proposition 2.6.

THEOREM 2.7. *Let S be a $\circ$-subsemigroup of F and let*

$$\mathcal{R}_S = \{A : a \in A \Rightarrow (\exists t \in S)(t(a) = 0)\} \quad \text{and} \quad \Phi_S = \{G \triangleleft F : G \cap S \neq \emptyset\}.$$

Then $\mathcal{R}_S$ is an elementary radical class. Moreover,

(i) *Φ_S satisfies (1) and (3) in the definition of a radical filter, and*
(ii) *if Φ_S is a radical filter then $\Phi_S = \Phi_{\mathcal{R}_S}$.*

A set Φ_S need not satisfy (2) in the definition of a radical filter. Let T be the set of Chebyshev polynomials (of the first kind) of odd degree. Then T can be viewed as a subset of F and is a semigroup with respect to $\circ$. By [4, Theorem 2.6], $\mathcal{R}_T$ is the class of odd torsion nil rings.

PROPOSITION 2.8. *In the notation just introduced, Φ_T is not a radical filter.*

PROOF. We have to show that Φ_T does not satisfy (2) of the definition of a radical filter. The standard reference for Chebyshev polynomials is the book of Rivlin [10]; here, we give a few facts to make the proof a bit more self-contained. The nth Chebyshev polynomial T_n has degree n and $T_n(x) = \cos n\theta$ on the open interval $(-1, 1)$, where $x = \cos\theta$ and $T_n(x)$ has n roots, all in $(-1, 1)$. For all m, n, we have $T_m \circ T_n = T_{mn}$. For even n, T_n has an x-free term and so cannot be treated as an element of F.

The principal ideal (T_3) is in Φ_T, so in particular, if Φ_T is to be a radical filter, $((T_3) * 10x)$ must be in Φ_T, that is, it must contain some T_n. This means that for some n, $T_n(10x)$ must be in (T_3). The roots of T_3 are 0 and $\pm\sqrt{3}/2$, and all elements of

(T_3) must have at least these roots. But all roots of Chebyshev polynomials are in the interval $(-1, 1)$, so $T_n(10\sqrt{3}/2) = T_n(5\sqrt{3}) \neq 0$ and thus $T_n(10x) \notin (T_3)$, that is, $T_n \notin ((T_3) * 10x)$. □

Thus, the relationship between a semigroup and the filter of the radical class it defines remains mysterious.

3. Other types of rings

All that we have done so far can be done just as well with algebras over a field K, with F replaced by the free algebra $xK[x]$. A periodic K-algebra is a periodic ring, so by [12, Theorem 3.4] if K has characteristic 0 there are no nonzero periodic K-algebras and all nontrivial elementary radical classes are contained in the radical class $\mathcal{N}_K$ of nil K-algebras. But the elementary radical subclasses of $\mathcal{N}$ are those of the form $\{A \in \mathcal{N} : A^+ \in \mathcal{A}\}$ for a radical class $\mathcal{A}$ of abelian groups, where A^+ is the additive group of A ([4], Theorem 4.1). As there no nontrivial radical classes of vector spaces, $\mathcal{N}_K$ has no proper elementary radical subclasses (and is the only nontrivial elementary radical class of K-algebras if K has characteristic 0). If K has prime characteristic, the elementary radical clases of periodic K-algebras are like those of rings, but all fields involved are algebraic extensions of K.

THEOREM 3.1.

(i) *If K is a field of characteristic 0, the only nontrivial radical filter of ideals of $xK[x]$ is $\{I : (\exists\, n)(x^n \in I)\}$, which corresponds to $\mathcal{N}_K$.*
(ii) *If K is a field of prime characteristic then $\{I : (\exists\, n)(x^n \in I)\}$ corresponding to $\mathcal{N}_K$ is the only radical filter associated with a class of nil algebras, and all other elementary radical classes of K-algebras consist of periodic algebras built from algebraic extensions of K.*

In any variety of not necessarily associative rings with the same one-generator free ring F as the class of associative rings (for example, alternative, power-associative and Jordan rings), there must be the same radical filters, and although in different settings the rings belonging to the radical classes corresponding to a given filter will sometimes be different, anything describable in terms of F alone, such as the identities which define a radical-semi-simple class, must be the same.

In such varieties (where 'nil' has its usual meaning but some circumspection is required in the definition of 'nilpotent'), elementary radical subclasses of the nil radical class are still additively determined: to prove the required version of Theorem 4.1 in [4], use [3, Corollary 5.6], which is based on a result from [13].

Theorem 13.2 of [9], which we have already used, and which shows among other things that periodic rings are torsion rings, was actually proved for power-associative rings. All periodic alternative rings are associative, but for Jordan rings there are nonassociative simple periodic Jordan rings ([9], Proposition 15.4). (If squeamish about 2-torsion, consider algebras over the subring of $\mathbb{Q}$ generated by 1/2.)

4. What about groups?

The group-theoretic analogue of the Ryabukhin correspondence is a natural subject for enquiry, and we shall give the details in this final section. We call a radical class of groups *elementary* if a group belongs to this class if and only if all its cyclic subgroups do so. We begin with a rather obvious result which is useful in what follows.

PROPOSITION 4.1. *For a multiplicative submonoid $\mathbb{M} \neq \{1\}$ of the positive integers, the following conditions are equivalent:*

(i) $mn \in \mathbb{M} \Rightarrow m, n \in \mathbb{M}$;
(ii) $\mathbb{M}$ *is generated by primes.*

We denote the identity element of any group by e and the cyclic subgroup generated by a group element g by $\langle g \rangle$. If an elementary radical class contains a group with an element of infinite order, it contains the infinite cyclic groups and hence all cyclic groups, and so all groups. Thus, we need only consider torsion groups. Write $o(g)$ for the order of an element $g \in G$.

THEOREM 4.2. *A radical class $\mathcal{R}$ of groups is elementary if and only if*

$$\mathcal{R} = \mathcal{R}_{\mathbb{M}} = \{G : g \in G \Rightarrow o(g) \in \mathbb{M}\}$$

for some monoid $\mathbb{M}$ generated by primes.

PROOF. Let $\mathbb{M}$ be the submonoid of the positive integers generated by a set E of primes. If $N \triangleleft G \in \mathcal{R}_{\mathbb{M}}$, then $o(g) \in \mathbb{M}$ for each $g \in G$ and $(gN)^{o(g)} = e$ so $o(gN)$, as a divisor of $o(g)$, is in $\mathbb{M}$, whence $G/N \in \mathcal{R}_{\mathbb{M}}$ and $\mathcal{R}_{\mathbb{M}}$ is homomorphically closed. If

$$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_\alpha \cdots$$

is a chain of normal subgroups of a group with each $N_\alpha \in \mathcal{R}_{\mathbb{M}}$, then any element y of $\bigcup_\alpha N_\alpha$ is in some N_β, so $o(y) \in \mathbb{M}$ and thus $\bigcup_\alpha N_\alpha \in \mathcal{R}_{\mathbb{M}}$. Finally, if $H \triangleleft G$, and H and G/H are both in $\mathcal{R}_{\mathbb{M}}$, let z be an element of G and let $m = o(zH)$. Then $m \in \mathbb{M}$ and $z^m \in H$, whence $o(z^m) \in \mathbb{M}$. Set $n = o(z^m)$. Then $z^{nm} = e$, so $o(z) \mid mn \in \mathbb{M}$. But then $o(z) \in \mathbb{M}$. It follows that $\mathcal{R}_{\mathbb{M}}$ is a radical class.

It is straightforward to show that a group is in $\mathcal{R}_{\mathbb{M}}$ if and only if all its cyclic subgroups are in $\mathcal{R}_{\mathbb{M}}$.

Conversely, for an elementary radical class $\mathcal{R}$, let $\mathbb{M} = \{o(g) : g \in G \in \mathcal{R}\}$. If $m, n \in \mathbb{M}$, let $m = o(g)$, $n = o(h)$ for elements g, h of groups in $\mathcal{R}$. Then $\langle g \rangle, \langle h \rangle \in \mathcal{R}$, so (by closure under extensions!) $\langle g \rangle \times \langle h \rangle \in \mathcal{R}$ and hence $mn = o(g, h) \in \mathbb{M}$, so $\mathbb{M}$ is a semigroup. In fact, $\mathbb{M}$ is a monoid, since $1 = o(e) \in \mathbb{M}$. If $k = \ell t \in \mathbb{M}$, let $k = o(w)$ for some $w \in H \in \mathcal{R}$. Then $o(w^\ell) = t$, so $t \in \mathbb{M}$, and $\mathbb{M}$ is therefore generated by primes. Now, clearly, $\mathcal{R} \subseteq \mathcal{R}_{\mathbb{M}}$, but if a group H is in $\mathcal{R}_{\mathbb{M}}$ all its cyclic subgroups are in $\mathcal{R}$ and so H itself is in $\mathcal{R}$. Hence $\mathcal{R} = \mathcal{R}_{\mathcal{M}}$. □

The free group on a single generator is an infinite cyclic group $\langle x\rangle$, which acts on groups by evaluation as in rings: x^n acting on g gives g^n. Analogous to $(0 * a)$,

$$(e * g) = \{x^n : g^n = e\} = \{x^n : o(g) \mid n\} = \langle x^{o(g)}\rangle.$$

Thus, we need to look at the set

$$\{\langle x^{o(g)}\rangle : g \in G \in \mathcal{R}_{\mathbb{M}}\} = \{\langle x^m\rangle : m \in \mathbb{M}\},$$

to which it is convenient to give the name $\Phi_{\mathbb{M}}$ rather than $\Phi_{\mathcal{R}}$.

The sets $\Phi_{\mathbb{M}}$ satisfy the analogues of (1), (2) and (3) in the definition of a radical filter. Only (3) presents any difficulty.

Let $\langle x^m\rangle$ be in $\Phi_{\mathbb{M}}$ and let $n \in \mathbb{Z}^+$ be such that $o(a\langle x^n\rangle) \in \mathbb{M}$ for each $a \in \langle x^m\rangle$. Then, in particular, $o(x^m\langle x^n\rangle) \in \mathbb{M}$. Let r be the lowest common multiple of m and n, with $r = r'm = r''n$. Then $(x^m)^{r'} = x^{mr'} = x^{nr''} \in \langle x^n\rangle$. If also $(x^m)^t = (x^n)^s$ for some $t, s \in \mathbb{Z}^+$, then $x^{mt} = x^{ns}$ so that $mt = ns$ is a common multiple of m and n, so $r'm = r \mid mt$, whence $r' \mid t$ and $r' = o(x^m\langle x^n\rangle) \in \mathbb{M}$. But then $n \mid r''n = r'm \in \mathbb{M}$, so $n \in \mathbb{M}$. This is (3).

Conversely, let Φ be a set of subgroups of $\langle x\rangle$ satisfying (1), (2) and (3), and $\mathbb{M}_\Phi = \{m \in \mathbb{Z}^+ : \langle x^m\rangle \in \Phi\}$. If $k \in \mathbb{M}_\Phi$ and $\ell \mid k$, then $\langle x^k\rangle \subseteq \langle x^\ell\rangle$. Since $\langle x^k\rangle$ is in Φ, (1) implies that $\langle x^\ell\rangle \in \Phi$, that is, $\ell \in \mathbb{M}_\Phi$ and so $\mathbb{M}_\Phi$ is 'closed under factors'. If $m, n \in \mathbb{M}_\Phi$, then $\langle x^m\rangle \in \Phi$, and for each $a \in \langle x^m\rangle$,

$$o(a\langle x^{mn}\rangle) \mid |\langle x^m\langle x^{mn}\rangle\rangle| = |\langle x^m\rangle/\langle x^{mn}\rangle| = m \in \mathbb{M}_\Phi,$$

so $o(a\langle x^{mn}\rangle) \in \mathbb{M}_\Phi$, whence $\langle x^{mn}\rangle \in \Phi$ by (3), that is, $mn \in \mathbb{M}_\Phi$.

It is easily seen that the correspondences between radical classes and monoids and between monoids and filters are bijective. Because all radical classes contain the group $\{e\}$ and all filters contain $\langle x\rangle$, monoids rather than semigroups are involved.

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