European Journal of Physics

PAPER

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To cite this article: Vitalie Chistol and Vasile Tronciu 2021 *Eur. J. Phys.* **42** 065002

IOP Publishing European Journal of Physics
Eur. J. Phys. **42** (2021) 065002 (9pp) https://doi.org/10.1088/1361-6404/ac1447

Graphical method of solving problems on bistability in physical systems

Vitalie Chistol and Vasile Tronciu*

Department of Physics, Technical University of Moldova, Stefan cel Mare av. 168, Chisinau MD-2004, Moldova

E-mail: vasile.tronciu@fiz.utm.md

Received 16 December 2020, revised 7 July 2021
Accepted for publication 14 July 2021
Published 27 August 2021

Abstract

Solving physics problems is often done superficially, without making an in-depth analysis of the results obtained, limiting itself only to determining the numerical values of some physical quantities. In order to analyze the results obtained after solving physics problems and to more deeply understand the essence of the phenomena that take place, a graphic representation of the obtained results is often necessary. In this paper, we present examples of solving different problems from physical systems for which the effect of bistability occurs. We use the graphical method to analyze the results obtained. We mention that the results presented in this paper are suited for teaching undergraduate students.

Keywords: graphical method, bistability, mechanics, undergraduate students, physical systems

(Some figures may appear in colour only in the online journal)

1. Introduction

The phenomenon of bistability is well-known in optics (optical bistability) [1], magnetism (magnetic bistability) [2], electricity (bistable circuits) [3] etc. In most cases the nature of this phenomenon is quantum. We propose in this paper an analysis of two examples of bistability in mechanics, and relatively simple explanation of this phenomenon. We use the graphical method to synthesize the results predicted by the theoretical models. Two examples could be useful for students to understand the phenomenon of bistability in physics.

The paper is structured as follows. We start in section 2 by describing rotation of cylinder with mobile piston. Section 3 shows bistable behavior of displacement of mercury in tubes.

*Author to whom any correspondence should be addressed.

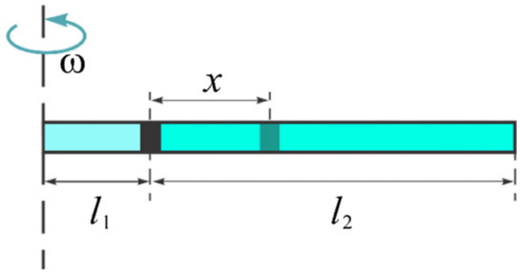

Figure 1. The scheme of the cylinder with mobile piston inside. By black, we show the initial position of piston when cylinder is static and by gray when the cylinder is rotating.

These two setups are well explained by graphical method. The summary and conclusions are given in section 4.

2. Rotation of cylinder with mobile piston

Let start our analysis considering a horizontal cylinder closed at both sides with a mobile piston of mass m inside. The piston can slide without friction along the cylinder.

The pressure of air in both sides of the cylinder is p_0, when cylinder is static in initial position. The initial distances from center of piston to the ends of cylinder are respectively l_1 and l_2 (see figure 1). We consider the follow question: at which angular velocity has to be rotated the cylinder so that the displacement of the piston from the equilibrium position is x?

To solve this problem, we find the forces acting to the piston. Two forces act to the piston from left and right parts of cylinder with the result force F_1 of the following form

$$F_1 = (p_2 - p_1)S, \tag{1}$$

where p_1 and p_2 are respectively, the air pressures on the left and right sides of piston during its rotation. S is the cross-sectional area of the cylinder. We mention that, during the rotation of the cylinder this force always is oriented to the axis of rotation.

In what follows, we consider the air in both sides of the cylinder subject to an isothermal process. Thus, we have

$$p_0 l_1 S = p_1(l_1 + x)S, \qquad p_0 l_2 S = p_2(l_2 - x)S.$$

Expressing p_1 and p_2 from the last equations and introducing them in (1), we obtain

$$F_1 = \left(\frac{p_0 l_2}{l_2 - x} - \frac{p_0 l_1}{l_1 + x}\right) S. \tag{2}$$

The force that gives to the piston centripetal acceleration can be written in the form

$$F_2 = m\omega^2(l_1 + x), \tag{3}$$

where ω is angular velocity.

When the piston is in equilibrium $F_1 = F_2$, i.e.

$$\left(\frac{p_0 l_2}{l_2 - x} - \frac{p_0 l_1}{l_1 + x}\right) S = m\omega^2(l_1 + x).$$

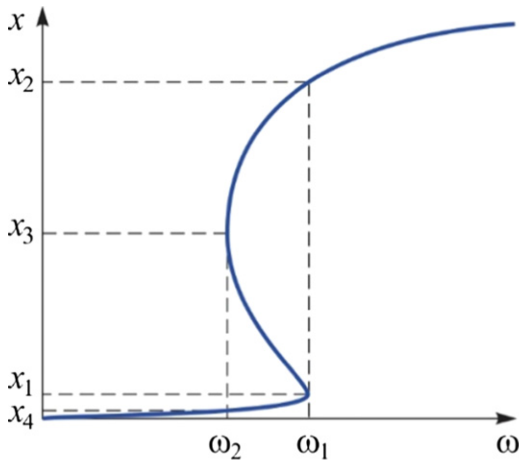

Figure 2. Schematic view of the dependence of the position of the piston x on the angular velocity of the cylinder ω.

From this equation we obtain the formula for ω

$$\omega = \sqrt{\frac{l_2(l_1 + x) - l_1(l_2 - x)}{m(l_2 - x)(l_1 + x)^2} p_0 S}. \tag{4}$$

From expression (4) one can see that for any value of $x < l_2$ we obtain a single value of ω. On the other hand, when we look to the dependence of $x(\omega)$ (see figure 2), we see that things are different. For a certain interval of values of the rotation speed, we can obtain three values of equilibrium position x.

Figure 2 shows that the displacement x of the piston increases with the increase of the angular velocity ω and for $\omega = \omega_1$ the piston jumps from position x_1 to x_2. Decreasing the angular velocity ω displacement x decreases, and for $\omega = \omega_2$ the piston again jumps from position x_3 to x_4. We observe that the piston cannot occupy positions in the range $x_1 < x < x_3$. Thus, we obtain an effect analogous to the bistability phenomenon in quantum physics.

To explain this phenomenon shown in figure 2, we plot in figure 3 the dependence of force $F_1(x)$ using equation (2) shown by red line (curve 1). We plotted also in this figure the force $F_2(x)$ for different values of ω. Thus, one can see that the slope of the lines 2, 3, and 4 depends on the angular velocity ω of the cylinder. For a certain angular velocity (line 2) there are three equilibrium states of the piston: x_5, x_6 and x_7.

When we start to rotate the cylinder at some angular velocity, the piston is in the equilibrium position x_5. For a small deviation of the piston to the right side (see figure 3), the air pressure force in the tube (curve 1) is higher than the centrifugal force (line 2) and the piston returns to the steady state. For a small deviation of the piston to the left side, the air pressure force in the tube is lower than the centrifugal force and the piston returns again to the steady state. Thus, the point x_5 is a stable equilibrium point. The characteristics of point x_7 are similar to those of point x_5. On the other hand, for state x_6, at a small deviation of the piston to the right, the air pressure force in the tube is lower than the centrifugal force and the piston will move away from the steady state, i.e. it moves to the steady state x_7. Opposite at a small deviation of the piston to the left, the air pressure force in the tube is higher than the centrifugal force and the

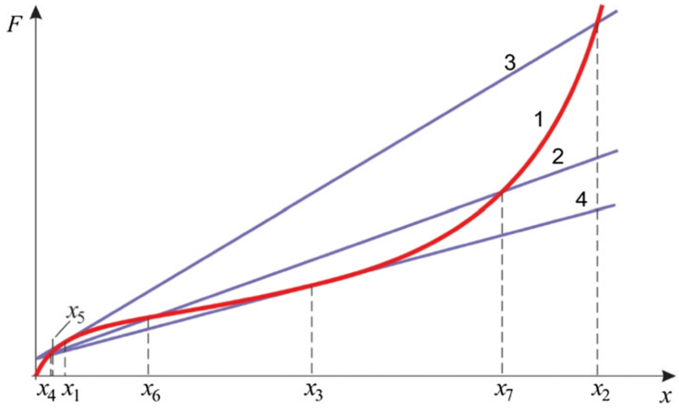

Figure 3. Forces acting on the piston in dependence on the displacement x. 1 displays the resulting force acting from the tube air (red curve); 2, 3, 4 show the centrifugal forces for different ω (blue lines).

piston will also move away from the steady state to the steady state x_5. Thus, point x_6 is an unstable equilibrium point.

Increasing the angular velocity ω, the slope of linear dependence (2) increases and for $\omega = \omega_1$ (see line 3 in figure 3) the piston has only two equilibrium states: x_1 and x_2. The stability of position x_2 does not differ from that of x_7, i.e. it is stable equilibrium position. The position x_1 is metastable equilibrium position [4] because the piston, being deviated slightly to the right, moves away from this position, and being deviated to the left returns back to it. At the angular velocity ω_1 the piston will be in metastable equilibrium in the position x_1. For a small increase of the angular velocity ω, the piston jumps to the position x_2. By further increase of angular velocity, x also increases. On the other hand, when we decrease the angular velocity, x also decreases. For $\omega = \omega_2$ (curve 4) the piston reaches the unstable position x_3. When we continue to decrease the rotational speed, the piston jumps to the position x_4. We mention that, someone could get the values of ω using x from the interval $x_1 < x < x_3$, in expression (4), but a wrong result will be obtained, because the piston cannot be in equilibrium in this domain.

To conclude, we consider what is predicted to happen when we increase the angular velocity ω (see figure 4). One can observe that for low angular velocity the displacement x increases monotonically with increase of ω along so called lower branch of bistability (blue curve). Note that, for $\omega = \omega_1$ the jump (switching) from x_1 to x_2 can be observed. Further on the dependence $x(\omega)$ follows the high branch of bistability. When considering the decrease of the angular velocity ω the system follows red curve in figure 4. Thus, we obtain a hysteresis loop in the dependence $x(\omega)$.

The results shown in figure 4 could be explained also by the competition of two forces acting on the piston: the centrifugal force and the resulting pressure force acting on the air in the tube. Initially, by increasing the rotational speed of the tube, the centrifugal force increases in proportion to the pressure and the piston moves away from the equilibrium position. At a rotational speed ω_1 the centrifugal force increases faster than the pressure force and the piston cannot be in equilibrium, so it moves away from the position $x = x_1$ without increasing the rotational speed. Approaching the end of the tube, the pressure force begins to increase more abruptly and reaches the moment when this force again becomes equal to the centrifugal one, and the piston stops in the position $x = x_2$. Further, increasing the rotational speed, both

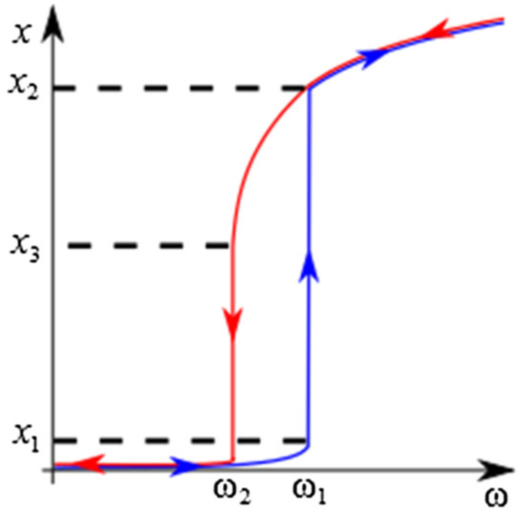

Figure 4. Schematic hysteresis loop in the dependence of $x(\omega)$.

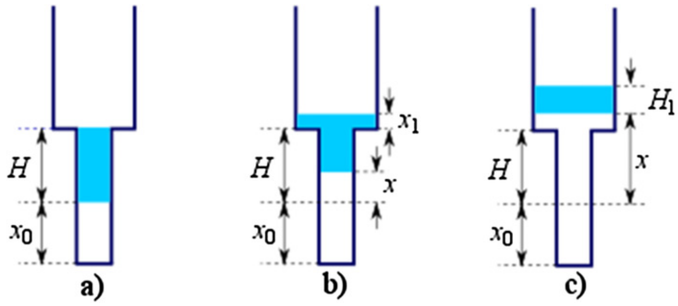

Figure 5. Heated air in tubes. (a) All mercury is in the lower tube. (b) Some of the mercury has passed into the upper tube. (c) All the mercury is in the upper tube.

forces increase proportionally and the piston gradually moves away from the initial position. By decreasing the rotation speed, the processes are repeated, only with the difference that in the opposite direction the switching will take place at a rotation speed $\omega_2 < \omega_1$.

3. Displacement of mercury in tubes

In what follows, we consider a similar case of hysteresis effect observed in a tube closed at one end, with a shape shown in figure 5. Inside the tube, there is a column of air with height x_0, separated from the atmosphere by a column of mercury with height H. T_0 is the initial air temperature in the tube. T is the temperature to vary. S_1 and S_2 are the cross-sectional areas of lower and upper connected parts, respectively. H_0 is the mercury height for normal atmospheric pressure. We will determine how much the air in the tube needs to be heated so that the mercury column move by x.

We start with the equation of state of the ideal gas for air in tube

$$\frac{pV}{T_0} = \frac{p_1 V_1}{T}, \tag{5}$$

where

$$p = \rho g (H + H_0), \qquad V = S_1 x_0.$$

For p_1 and V_1 when $x \leqslant H$ we can write expressions

$$p_1 = \rho g (H - x + x_1 + H_0), \qquad V_1 = S_1 (x_0 + x).$$

From figure 5(b) we observe that

$$S_1 x = S_2 x_1, \quad \text{or} \quad x_1 = \frac{S_1}{S_2} x = \alpha x, \quad \text{where } \alpha = \frac{S_1}{S_2}.$$

In this case $p_1 = \rho g (H + H_0 - (1 - \alpha) x)$.

Taking into account the above formulas from (5) we obtain

$$\frac{\rho g (H + H_0) S_1 x_0}{T_0} = \frac{\rho g (H + H_0 - (1 - \alpha) x) S_1 (x_0 + x)}{T_1}.$$

After some simple transformations we obtain

$$T_1 = T_0 \frac{(H + H_0 - (1 - \alpha) x)(x_0 + x)}{(H + H_0) x_0}. \tag{6}$$

For the case $x > H$ of high temperature we write the following expressions (see figure 5(c))

$$V_1 = S_1 (H + x_0) + S_2 (x - H),$$

$$P_1 = \rho g (H_0 + H_1) = \rho g (H_0 + \alpha H).$$

Using these formulas from (5) we obtain

$$T_2 = T_0 \frac{\rho g (H_0 + \alpha H) [S_1 (H + x_0) + S_2 (x - H)]}{\rho g (H_0 + H) S_1 x_0},$$

and after simple transformations one can write

$$T_2 = T_0 \frac{(H_0 + \alpha H) \left[H + x_0 + 1/\alpha (x - H)\right]}{(H_0 + H) x_0}. \tag{7}$$

Figure 6 shows the dependence of displacement on the temperature $x(T)$ using expressions (6)-blue curve 1 and (7)-red line 2. From this figure one observe, that at the temperature T' the displacement of the lower surface of the mercury is x'. For $T > T'$ the equation (6) has no solutions. This means that for $T > T'$ all the mercury from the surface tube S_1 passes by jumping into the surface tube S_2, and the displacement of the mercury will be $x > x_2$.

To explain this jump, we investigate the balance of the lower surface of mercury. To this surface acts the pressure p_1 from the mercury and the air in the atmosphere, also the pressure p_2 from the air in the tube. Mercury is in equilibrium when these pressures are equal $p_1 = p_2$. The pressures p_1 and p_2 have the following forms

$$p_1 = \begin{cases} \rho g (H_0 + H - x) & \text{for } x \leqslant H \\ \rho g (H_0 + H_1) = \rho g (H_0 + \alpha H) & \text{for } x > H, \end{cases} \tag{8}$$

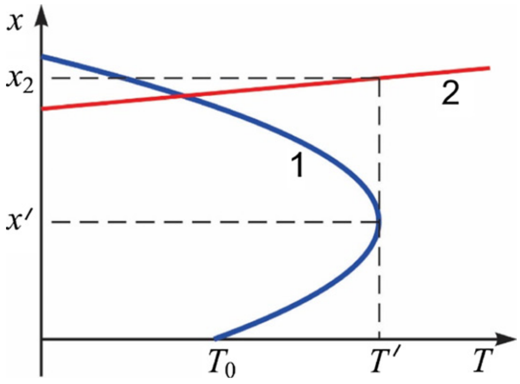

Figure 6. Dependence of mercury column displacement x on the air temperature in the tube. 1-for $x \leqslant H$ and 2-for $x > H$.

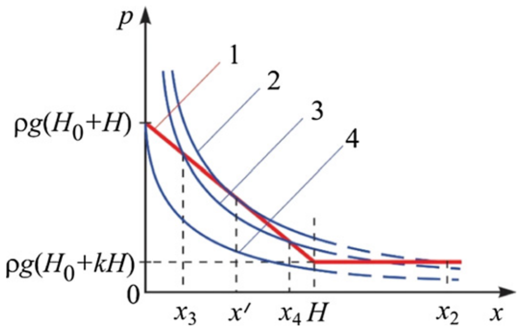

Figure 7. Dependence of lower surface pressure of the mercury on the displacement of the air column in the tube x. Red (thick) 1-pressure p_1. Blue (thin) 2, 3, 4-pressure p_2 for different temperatures (2-$T = T'$; 3-$T = T_1$; 4-$T = T_0$).

$$p_2 = \frac{\nu RT}{S_1\,(x_0 + x)} \quad \text{for } x \leqslant H. \tag{9}$$

Figure 7 shows these dependences on displacements i.e. $p_1(x)$ and $p_2(x)$ [5, 6]. We see that for $T = T_0$, in the region $x \leqslant H$, the curve $p_2(x)$ intersects the line $p_1(x)$ at $x = 0$. For temperature $T = T_1$ the dependencies $p_1(x)$ and $p_2(x)$ in the region $x \leqslant H$ have two points of intersection x_3 and x_4. One can easy observe that for $x = x_3$ the equilibrium of mercury is stable, and for $x = x_4$ is unstable. When $T = T'$ the dependencies $p_1(x)$ and $p_2(x)$ have only one intersection point x' in the region $x \leqslant H$ and an intersection point x_2 in the region $x > H$. For a small increase of temperature the dependencies $p_1(x)$ and $p_2(x)$ no longer have intersection points in the region $x \leqslant H$, and the mercury will move from the unstable equilibrium position x' to any stable equilibrium position $x > x_2$.

Figure 8 shows the dependence of displacement on temperature $x(T)$. We see that as long as the temperature increases, the displacement of lower layer of mercury surface also increases (curve 1) calculated from expression (6). At temperature $T = T'$ the displacement increases by jumping from x' to x_2. Further increasing the temperature, the displacement follows

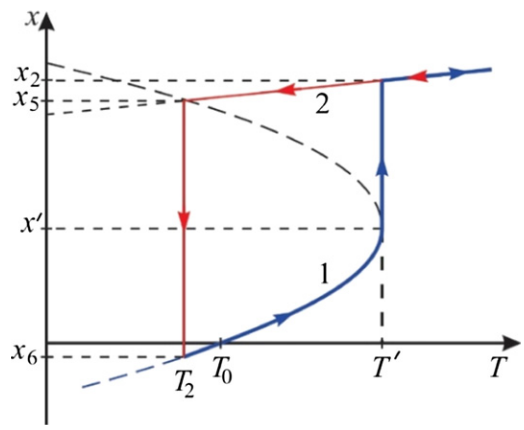

Figure 8. Hysteresis loop of system shown in figure 5.

line 2 calculated from expression (7). On the other hand, decreasing the temperature T leads to a decrease of mercury displacement. However, we observe the jump from position x_5 to x_6 at temperature $T = T_2$. We mention that, the temperature T_2 can be lower or higher than T_0. It depends on the choice of x_0, H and k. In this case, x_6 can be positive or negative. Finally, if it is necessary to calculate at which temperature the air in the tube must be heated, so that the displacement of the mercury has a value in the range $x' < x < x_5$ without taking into account the effect of bistability, we could obtain a wrong result.

Finally, we give a physical interpretation of processes that can take place in tube of figure 5 and explain the figure 8 hysteresis. Suppose that, the temperature of air in tube is constant and somehow forces the mercury column to rise upwards with a very small value of displacement x. Then both the air pressure in the tube p_2 and the pressure of the mercury column p_1 will decrease, since the mercury passes into the tube with the larger section and the height of its column decreases. However, these decreases in pressure will not be proportional one to other, but p_2 will decrease more than p_1. Since the displacement is small the mercury will return to its equilibrium position. However, for higher values of displacement x it may happen that p_2 decreases less than $p_{1.}$ Then the mercury will continue to move away from the equilibrium position until all the mercury passes into the tube with the larger section. As the mercury continues to move to high section tube p_1 remains constant, and p_2 continues to decrease until these two pressures equalize each other and a new equilibrium position of mercury is established. Thus, if the initial temperature of the gas is T_0 then with increasing temperature it increases displacement x (see figure 8). At temperature T' the mercury passes by jumping from position x' to x_2. By decreasing the temperature, the processes are repeated in the opposite direction, and the return of mercury to the tube with the smaller section will take place at another temperature $T_2 < T'$.

Finally, we mention that in most textbooks, that explain the phenomenon of optical bistability, the hysteresis is described more qualitatively, without a mathematical description. There are cases when the mathematical description of the hysteresis loop does not even exist for example the hysteresis loop of the ferroelectrics. At the same time, it is well known that the nature of superconductivity is not yet fully revealed. However, the red curves in figures 4 and 8 are very similar to those obtained in the case of superconductivity. In examples proposed for examination in this paper, two forces compete that lead to the appearance of the bistability effect. On the other hand, similar examples can appear in the case of superconductivity, where two effects may compete as well. Thus, the proposed approach could serve for students as an example of solving the problems and as an alternative to solve some theoretical problems related to hysteresis loops, and phase transitions of different systems. From the mathematical description of the forces acting in the system (equations (2) and (3)) of section 2 and equations (6) and (7)) of section 3) it is very difficult even sometimes impossible to come to the idea of bistability or hysteresis. Only the graphical representation (see figures 3 and 7) can lead to the idea of bistability and can serve as a model for trying to explain this effect. Based on graphical method the students can easily avoid the mistakes solving different problems from fields where nonlinearities and hysteresis are present. Sometimes is happens that, only using graphical method all solutions of hysteresis can be found either in electromagnetism, optics or other field of student study.

4. Conclusions

In conclusion, we mention that the graphical method presented in this paper is a very important tool for understanding the effect of bistability in physical systems. The results presented can be used to significantly improve the understanding of this phenomenon by undergraduate students, and for explaining the physical processes that take place. On the other hand, we denote that in the case of non-linear dependencies between physical quantities, we have to take into account that the effect of bistability can occur. Thus, to avoid mistakes in solving physics problems it is necessary to consider this phenomena. We believe that our work provides a good basis for future, more detailed studies of bistability in physical systems. The presented results may be used to significantly improve the understanding of phenomenon of bistability by undergraduate students.

Acknowledgments

Authors acknowledge the support from the Project 20.800 09.5007.08.

ORCID iDs

Vitalie Chistol https://orcid.org/0000-0002-4761-5892
Vasile Tronciu https://orcid.org/0000-0002-9164-2249

References

[1] Gibbs H 1985 *Optical Bistability: Controlling Light with Light* (New York: Academic) p 484
[2] Kahn O 2013 *Magnetism: A Supramolecular Function* vol 6 (Berlin: Springer) p 660
[3] Haraoubiais B 2019 *Nonlinear Electronics 2: Flip-Flops, ADC, DAC and PLL* (Amsterdam: Elsevier) p 316
[4] Chistol V, Pârţac C and Ungureanu N 2005 Applying physics problems to students technological education (in Romanian) *The Technical-Scientific Conf. Students and Doctoral Students Dedicated to the Year of Physics* (17 Noiembrie 2005) (Chişinău: UTM) pp 47–8
[5] Chistol V 2010 On the graphical method to solve problems in physics (in Romanian) *Cygnus* **1** 23–7
[6] Epstein V 2007 From simple to complicated (in Russian) *Kvant* **3** 34–6